A Long Exact Sequence in Symmetry Breaking

order parameter constraints, defect anomaly-matching, and higher Berry phases

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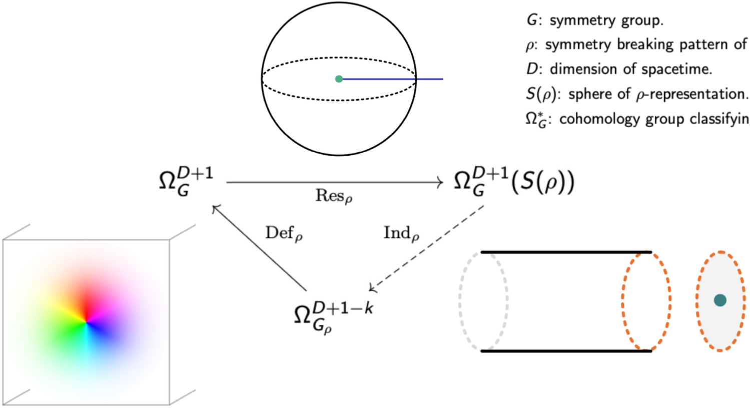
Outline

End Goal: Symmetry Breaking Long Exact Sequence



$$\begin{tikzcd}
\Omega_G^{D+1} \arrow[rr, "\mathrm{Res}_\rho"'] & & \Omega_G^{D+1}(S(\rho)) \arrow[dl, dashed, "\mathrm{Ind}_\rho"'] \\
& \Omega_{G_\rho}^{D+1-k} \arrow[ul, "\mathrm{Def}_\rho"] &
\end{tikzcd}$$

G: symmetry group.

ρ: symmetry breaking pattern of dimension k

D: dimension of spacetime.

$S(\rho)$: sphere of ρ-representation.

Ω^*_G: cohomology group classifying G-SPTs.

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- Bulk-boundary: β is a $D+1$-dim'l SPT and Z is a boundary theory of β.
- Mathematical classification: the anomaly $\beta \in \Omega_G^{D+1}$ lives in a cohomology class (more on this in the math section) [Kap14; Kap+15; FH21].

Families of Anomalies

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Definition

A theory is *ρ-gappable* if there are order parameters $(\mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_k)$, transforming in ρ under G, such that $H_{(c_1,...,c_n)} = H_0 + \sum_j c_j \int d^D x \mathcal{O}_j(x)$ has a gapped, nondegenerate ground state for large enough radius $R = \sum_j |c_j|^2$.

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- e.g.: Consider a $3+1$D Dirac fermion ψ with anomalous $G = U(1)_L$ symmetry. It is ρ-gappable for $\rho = \underline{1}$, given by the Dirac mass terms $(\mathcal{O}_1 = \bar{\psi}\psi, \mathcal{O}_2 = i\bar{\psi}\gamma^5\psi)$.

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- $\mathrm{Res}_\rho(\beta)$ is the obstruction to gapping Z over $S(\rho)$:

Z is ρ-gappable if and only if $\mathrm{Res}_\rho(\beta) = 0$.

Example: $2+1$D Majorana Fermion

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- Take $\rho = \rho_0 \oplus \rho_0$. Is the theory ρ-gappable? That is, can we find two T-odd operators $\mathcal{O}_1$, $\mathcal{O}_2$ such that

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- We claim that the answer is no! There are no operators $\mathcal{O}_1, \mathcal{O}_2$ that can make this happen!

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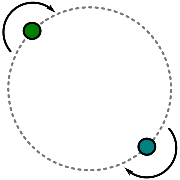
- Clearly this is not gapped at $\theta = \pm 3\pi/4$ where $m = 0$. Going from $m < 0$ to $m > 0$ pumps a $p + ip$ superconductor. Going from $m > 0$ to $m < 0$ pumps a $p - ip$.



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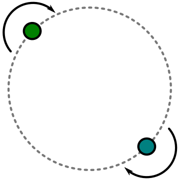
- Claim: The number of $p \pm ip$ pumped mod 2 across half an arc is an invariant.



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- Let's check this on anomalies. The Majorana fermion's anomaly is $1 \in \Omega^4_{\mathrm{Pin}^+} = \mathbb{Z}_{16}$ [Wit16]. $\Omega^4_{\mathrm{Pin}^+}(S^1) = \mathbb{Z}_2$ counts the number of $p \pm ip$ mod 2 across the arc.

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- Consequence I: For any choices of T-odd $\mathcal{O}_1, \mathcal{O}_2$, the number of $p \pm ip$'s pumped across half an arc is odd.
- Consequence II: The $2+1$D Majorana fermion is not ρ-gappable.

Recap I

$$\Omega_G^{D+1} \xrightarrow[\mathrm{Res}_\rho]{} \Omega_G^{D+1}(S(\rho))$$

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$$\phi = (v_1 x_1 + \ldots + v_k x_k) / \sqrt{\sum_i x_i^2}$$

for large $R = \sqrt{\sum_i x_i^2}$, where the v_i form an orthonormal basis of V_ρ.

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- Examples: domain walls ($k = 1$), vortices ($k = 2$), and hedgehogs ($k = 3$).

Defect Anomaly Matching

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- The anomaly matching condition [HKT20b]:

$$\mathrm{Def}_\rho(\alpha) = \beta.$$

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$$\beta = \frac{1}{6}(c_1)^3 - \frac{1}{24}c_1 p_1(TX),$$

where p_1 is the first Pontryagin number and X is the $6D$ test manifold with a principal $U(1)_L$ bundle P. [2]

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- The defect anomaly polynomial is
$$\alpha = \frac{1}{8}(c_1)^2 - \frac{1}{24}p_1(TY), \tag{1}$$
where Y is the zero section of a generic section $s\colon X \to E_\rho = P \times_{U(1)_L} V_\rho$.

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- Plugging in, we get

$$\mathrm{Def}_\rho(\alpha) = \frac{1}{6}(c_1)^3 - \frac{1}{24}c_1 p_1(TX) = \beta.$$

Recap II

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- β is the anomaly of the defect system created via the ρ-gapping.

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- Question: What is the ambiguity in the defect anomaly map?
- The defect comes from a ρ-gapping, which assigns a nondegenerate ground state to each point on the sphere $S(\rho)$. This invertible family is not typically free of G-anomalies, but it is when $\beta = 0$. Therefore the ρ-gapping defines a D-dim'l SPT class
$$\gamma \in \Omega^D_G(S(\rho)).$$

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- Recall our ρ-gapping Hamiltonian:

$$H_{(c_1,\ldots,c_n)} = H_0 + \sum_j c_j \int d^D x \mathcal{O}_j(x)$$

Since Z is anomaly-free, let's assume H_0 has a symmetric non-degenerate ground state. $H_{c_1,\cdots,c_k}$ is also gapped for large $R = \sum_j |c_j|^2$.

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  Since Z is anomaly-free, let's assume H_0 has a symmetric non-degenerate ground state. $H_{c_1,\cdots,c_k}$ is also gapped for large $R = \sum_j |c_j|^2$.

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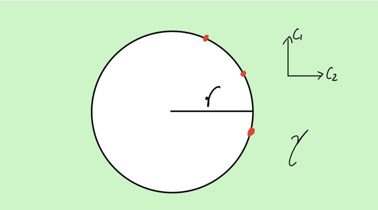

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- Viewing the S^1 parameter theory at $r=1$ as the boundary of γ, we see that when we adiabatically vary the S^1 parameter ϕ, we pump a quantized charge to the boundary [Tho83].

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- The first group counts the charges pumped when we vary S^1 parameter ϕ; the latter computes the $U(1)$ charge of the ϕ-vortex.

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- We have a sequence of maps:

$$\Omega_G^D(S(\rho)) \xrightarrow{\mathrm{Ind}_\rho} \Omega_{G_\rho}^{D+1-k} \xrightarrow{\mathrm{Def}_\rho} \Omega_G^{D+1} \xrightarrow{\mathrm{Res}_\rho} \Omega_G^{D+1}(S(\rho))$$

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Part II: Math and Applications

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SBLES as Induced by a Map of Spectra

$$\begin{tikzcd}
MT\mathrm{Spin} \wedge BG \arrow[dr] & & MT\mathrm{Spin} \wedge S(\rho)^{\rho^* \xi} \arrow[ll] \\
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Example SBLES and Map of Spectra

Running example:

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- $I_{\mathbb{Z}}^4(MT\mathrm{Pin}^+) \cong \mathbb{Z}/16$ (2 + 1D Majorana)
- $\pi_2(MT\mathrm{Pin}^+) \cong \Omega_2^{\mathrm{Pin}^+} \cong \mathbb{Z}/2$ (Klein bottle)

Madsen-Tillman Spectra and Anomalies—Takeaway

- Fix a stable tangential structure $\xi \colon B \to BO$ (e.g. $\xi \colon B\mathrm{Pin}^+ \to BO$ for fermions with $T^2 = (-1)^F$)
- "Definition": The *Madsen-Tillman spectrum* $MT\xi$ is the Thom spectrum of the inverse of ξ, written $B^{-\xi}$. (e.g. $MT\mathrm{Pin}^+$)
- Idea: The Madsen-Tillman spectrum $MT\xi$ is such that

Ansatz ([FH21])

$$\left\{\begin{array}{l}\text{anomaly groups}\\ \text{for } D\text{-dim'l theories}\\ \text{with symmetry } (B,\xi)\end{array}\right\} \cong I_{\mathbb{Z}}^{D+2}(MT\xi) = \Omega_{\xi}^{D+1}$$

Theorem (Pontrjagin-Thom)

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Proposition

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$$S(\rho)^{p^*\xi} \longrightarrow B^{-\xi} \xrightarrow{\mathrm{sm}_\rho} B^{-\xi+\rho}.$$

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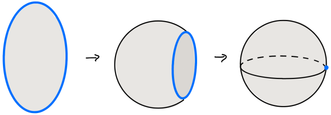

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- $MT\mathrm{Spin} \wedge \Sigma^{\infty-1}_{+}\mathbb{R}P^2 \to MT\mathrm{Pin}^- \xrightarrow{\mathrm{sm}_{2\sigma}} \Sigma^2 MT\mathrm{Pin}^+$ [KT90]

Review: SBLES as Induced by a Map of Spectra

$$\begin{tikzcd}
MT\mathrm{Spin} \wedge BG \arrow[dr, "\mathrm{sm}_\rho"] & & MT\mathrm{Spin} \wedge S(\rho)^{p^*\xi} \arrow[ll] \\
& MT\mathrm{Spin} \wedge BG^\rho \arrow[ur, dashed] &
\end{tikzcd}$$

$I_{\mathbb{Z}}^*$ cohomology $\rightsquigarrow$

$$\begin{tikzcd}
\Omega_G^{D+1} \arrow[rr, "\mathrm{Res}_\rho"'] & & \Omega_{G,f}^{D+1}(S(\rho)) \arrow[dl, dashed, "Ind"] \\
& \Omega_{G^\rho,f}^{D+1-k} \arrow[ul, "Def"] &
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Review: SBLES as Induced by a Map of Spectra

- Recall: Anomalies are classified by $\Omega_{\xi}^{D+1} = I_{\mathbb{Z}}^{D+2}(MT\xi)$

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- Idea: Fiber sequence of spectra $\overset{\text{take cohomology}}{\rightsquigarrow}$ long exact sequence

$$\begin{tikzcd}
MT\mathrm{Spin} \wedge BG \arrow[dr, "\mathrm{sm}_\rho"] & & MT\mathrm{Spin} \wedge S(\rho)^{p^*\xi} \arrow[ll] & \overset{I_{\mathbb{Z}}^* \text{ cohomology}}{\rightsquigarrow} & \Omega_G^{D+1} \arrow[rr, "\mathrm{Res}_\rho"'] & & \Omega_{G,f}^{D+1}(S(\rho)) \arrow[dl, dashed, "Ind"] \\
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Part II: Math and Applications

1. How to mathematically derive the SBLES
2. How to apply it
    1. Computing Def_ρ to perform anomaly matching
    2. Computing anomaly groups

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$$\mathrm{Def}_\sigma : \mathbb{Z}/8 \oplus \mathbb{Z} \longrightarrow \mathbb{Z}/16$$

- Turns out, this is $(a, b) \mapsto b - 2a$, where b tracks the gravitational anomaly of the defect theory and a tracks the internal $\mathbb{Z}/2$ anomaly [HKT20b]

Spin $\times$ $\mathbb{Z}/2 \rightsquigarrow \mathrm{Pin}^+$ Defect Matching Maps

$$\begin{array}{cccc}
 & \Omega^{*-1}_{\mathrm{Spin}\times\mathbb{Z}/2} & & \Omega^{*}_{\mathrm{Pin}^+} \\
-1 & 0 & & 0 \\
0 & \mathbb{Z} & \xrightarrow{?} & \mathbb{Z}/2 \\
1 & 0 & & 0 \\
2 & (\mathbb{Z}/2)^2 & \xrightarrow{?} & \mathbb{Z}/2 \\
3 & (\mathbb{Z}/2)^2 & \xrightarrow{?} & \mathbb{Z}/2 \\
4 & \mathbb{Z}\oplus\mathbb{Z}/8 & \xrightarrow{?} & \mathbb{Z}/16
\end{array}$$

$\mathrm{Spin} \times \mathbb{Z}/2 \rightsquigarrow \mathrm{Pin}^+$ SBLES

$$\begin{tikzcd}
 & \Omega^{*-1}_{\mathrm{Spin}\times\mathbb{Z}/2} & \Omega^{*}_{\mathrm{Pin}^+} & \Omega^{*}_{\mathrm{Spin}} \\
-1 & 0 & 0 & \mathbb{Z} \arrow[dll] \\
0 & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 & 0 \\
1 & 0 & 0 & \mathbb{Z}/2 \arrow[dll] \\
2 & (\mathbb{Z}/2)^2 \arrow[r] & \mathbb{Z}/2 & \mathbb{Z}/2 \arrow[dll] \\
3 & (\mathbb{Z}/2)^2 \arrow[r] & \mathbb{Z}/2 & \mathbb{Z} \arrow[dll] \\
4 & \mathbb{Z} \oplus \mathbb{Z}/8 \arrow[r] & \mathbb{Z}/16 & 0
\end{tikzcd}$$

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- Long exact sequences can aid in anomaly group computations (solving extension problems)
- Recall: *Smith maps* sm_ρ are dual to defect maps Def_ρ
- *Bordism groups* are dual to *anomaly groups*
- Example: σ-twisted bordism of $\mathbb{R}P^2$
- Other examples: [Deb+23] studying the Swampland Cobordism Conjecture

Pin$^{\pm}$ Long Exact Sequence in Bordism

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- Consider $\rho = 2\sigma$ and fermionic theories with internal time reversal [KT90]:

$$\mathrm{sm}_{2\sigma} \colon \Omega_d^{\mathrm{Pin}^-} \longrightarrow \Omega_{d-2}^{\mathrm{Pin}^+}.$$

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- The fiber sequence inducing this is

$$MT\mathrm{Spin} \wedge \Sigma^{-1}\mathbb{R}P^2 \longrightarrow MT\mathrm{Pin}^+ \xrightarrow{\mathrm{sm}_{2\sigma}} \Sigma^2 MT\mathrm{Pin}^-$$

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$$MT\mathrm{Spin} \wedge \Sigma^{-1}\mathbb{R}P^2 \longrightarrow MT\mathrm{Pin}^+ \xrightarrow{\mathrm{sm}_{2\sigma}} \Sigma^2 MT\mathrm{Pin}^-$$

- To fill in the LES, we need to compute

$$\pi_*(MT\mathrm{Spin} \wedge \Sigma^{-1}\mathbb{R}P^2) \cong \widetilde{\Omega}_{*+1}^{\mathrm{Spin}}(\mathbb{R}P^2) \cong \Omega_*^{\mathrm{Spin}}(\mathbb{R}P^1, \sigma)$$

Pin$^\pm$ Long Exact Sequence in Bordism—[KT90] Computation

$\mathrm{Pin}^\pm$ Long Exact Sequence in Bordism—[KT90] Computation

- Kirby-Taylor observed [KT90] that the degree-two map

$$\mathbb{S} \xrightarrow{\cdot 2} \mathbb{S} \longrightarrow \Sigma_+^{\infty-1}\mathbb{R}P^2$$

induces ·2 on spin bordism (dual to anomaly groups):

$$\Omega_*^{\mathrm{Spin}} \xrightarrow{\cdot 2} \Omega_*^{\mathrm{Spin}} \longrightarrow \Omega_*^{\mathrm{Spin}}(\mathbb{R}P^1, \sigma).$$

LES Partially Determining $\Omega_*^{\mathrm{Spin}}(\mathbb{R}P^1, \sigma)$

$*$	Ω_*^{Spin}	Ω_*^{Spin}	$\Omega_*^{\mathrm{Spin}}(\mathbb{R}P^1, \sigma)$
5	0	0	
4	$\mathbb{Z}$	$\mathbb{Z}$	
3	0	0	
2	$\mathbb{Z}/2$	$\mathbb{Z}/2$	
1	$\mathbb{Z}/2$	$\mathbb{Z}/2$	
0	$\mathbb{Z}$	$\mathbb{Z}$	

LES Partially Determining $\Omega_*^{\mathrm{Spin}}(\mathbb{R}P^1, \sigma)$

$$\begin{tikzcd}
* & \Omega_*^{\mathrm{Spin}} & \Omega_*^{\mathrm{Spin}} & \Omega_*^{\mathrm{Spin}}(\mathbb{R}P^1, \sigma) \\
5 & 0 & 0 & {\color{blue}0} \\
4 & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & {\color{blue}\mathbb{Z}/2} \\
3 & 0 & 0 & {\color{blue}\mathbb{Z}/2} \arrow[dll, rounded corners] \\
2 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \arrow[r] & {\color{red}A} \arrow[dll, rounded corners] \\
1 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \arrow[r] & {\color{blue}\mathbb{Z}/2} \arrow[dll, rounded corners] \\
0 & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & {\color{blue}\mathbb{Z}/2}
\end{tikzcd}$$

Resolving the Extension Question with the Smith LES

$*$	$\Omega_*^{\mathrm{Spin}}(\mathbb{R}P^1, \sigma)$	$\Omega_*^{\mathrm{Pin}^-}$	$\Omega_{*-2}^{\mathrm{Pin}^+}$
6		$\mathbb{Z}/16$	$\mathbb{Z}/16$
5		0	$\mathbb{Z}/2$
4		0	$\mathbb{Z}/2$
3		0	0
2		$\mathbb{Z}/8$	$\mathbb{Z}/2$
1		$\mathbb{Z}/2$	0
0		$\mathbb{Z}/2$	0

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$$\begin{tikzcd}
* & \Omega_*^{\mathrm{Spin}}(\mathbb{R}P^1, \sigma) & \Omega_*^{\mathrm{Pin}^-} & \Omega_{*-2}^{\mathrm{Pin}^+} \\
6 & 0 & \mathbb{Z}/16 \arrow[r] & \mathbb{Z}/16 \\
5 & 0 & 0 & \mathbb{Z}/2 \arrow[dll, out=-30, in=150] \\
4 & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 \arrow[dll, out=-30, in=150] \\
3 & \mathbb{Z}/2 & 0 & 0 \\
2 & \mathbb{Z}/4 \arrow[r] & \mathbb{Z}/8 \arrow[r] & \mathbb{Z}/2 \\
1 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 & 0 \\
0 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 & 0
\end{tikzcd}$$

[Optional:] Twisted Tangential Structures and Shearing

Definition

Let $V \to X$ be a virtual bundle. An (X, V)-*twisted spin structure* on a vector bundle $E \to M$ is

- a map $f\colon M \to X$
- a spin structure on $E \oplus f^*V$

- Manifolds with (X, V)-twisted spin structures live in $\pi_*(MT\mathrm{Spin} \wedge X^{V-r})$.

Examples

- Pin^+-structures $\leftrightarrow$ $(B\mathbb{Z}/2, \sigma)$-twisted spin structures
  - check $w_2(E) = 0 \iff E \oplus 3\mathrm{Det}(E)$ is spin
  - $MT\mathrm{Pin}^+ \simeq MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)^{3\sigma-3}$
- Spin^c-structures $\leftrightarrow$ $(BU(1), \gamma)$-twisted spin structures
  - $MT\mathrm{Spin}^c \simeq MT\mathrm{Spin} \wedge BU(1)^{\gamma-1}$

Application 2: Computing Anomaly Groups

Thanks for coming!

Examples of Periodic Families

Smith homomorphisms often occur in periodic families:

- 1-periodic family ([CF64]):

$$\Omega_d^{O\times\mathbb{Z}/2} \xrightarrow{\mathrm{sm}_\sigma} \Omega_{d-1}^{O\times\mathbb{Z}/2} \xrightarrow{\mathrm{sm}_\sigma} \Omega_{d-2}^{O\times\mathbb{Z}/2} \longrightarrow \dots$$

- 2-periodic family ([KT90; Sto88]):

$$\Omega_d^{\mathrm{Spin}} \xrightarrow{\mathrm{sm}_\gamma} \Omega_{d-2}^{\mathrm{Spin}^c} \xrightarrow{\mathrm{sm}_\gamma} \Omega_{d-4}^{\mathrm{Spin}} \xrightarrow{\mathrm{sm}_\gamma} \Omega_{d-6}^{\mathrm{Spin}^c} \longrightarrow \dots$$

- 4-periodic family ([HKT20b; BC18; Sto88; KT90; Pet68]):

$$\Omega_d^{\mathrm{Spin}\times\mathbb{Z}/2} \xrightarrow{\mathrm{sm}_\sigma} \Omega_{d-1}^{\mathrm{Pin}^-} \xrightarrow{\mathrm{sm}_\sigma} \Omega_{d-2}^{\mathrm{Spin}\times_{\mathbb{Z}/2}\mathbb{Z}/4} \xrightarrow{\mathrm{sm}_\sigma} \Omega_{d-3}^{\mathrm{Pin}^+} \xrightarrow{\mathrm{sm}_\sigma} \Omega_{d-4}^{\mathrm{Spin}\times\mathbb{Z}/2} \longrightarrow \dots$$

Untwisting

Idea: Let ρ be the k-dim'l twisting datum.

- periodic Smith families (with period n) occur when $n\rho$ is appropriately oriented.
- in that case, the spectrum untwists:

$$MTH \wedge X^{n\rho} \simeq MTH \wedge \Sigma^{kn} X$$

The Spin Case

There is an isomorphism of $MT\mathrm{Spin}$-modules

$$MT\mathrm{Spin} \wedge X^{n\rho} \simeq MT\mathrm{Spin} \wedge \Sigma^{nk} X$$

if and only if $n\rho$ has a spin structure.

- The order of the image of $\rho \in [X, BO]$ under the homomorphism $[X, BO] \to [X, BO/B\mathrm{Spin}]$ determines periodicity

Untwisting

Idea: When $n\rho$ is appropriately oriented, the spectrum untwists:

$$MTH \wedge X^{n\rho} \simeq MTH \wedge \Sigma^{kn} X$$

Examples

- $n = 1$: $X = B\mathbb{Z}/2$; no orientation condition for σ
  - $MTO \wedge (B\mathbb{Z}/2)^{\sigma}_{+} \simeq MTO \wedge \Sigma(B\mathbb{Z}/2)_{+}$
- $n = 2$: $X = BU(1)$; 2γ is spin
  - check: for any complex vector bundle E, E is oriented, and $2E$ is spin
  - $MT\mathrm{Spin} \wedge BU(1)^{2\gamma} \simeq MT\mathrm{Spin} \wedge \Sigma^4 BU(1)$
- $n = 4$: $X = B\mathbb{Z}/2$; 4σ is spin
  - check: for any real bundle E, $2E$ is oriented, and $4E$ is spin
  - $MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)^{4\sigma}_{+} \simeq MT\mathrm{Spin} \wedge \Sigma^4 B\mathbb{Z}/2.$

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