Abelian duality in topological field theories

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Outline

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- In the language of higher form gauge theories, We can view sigma model as 0-form gauge theories, standard gauge theories are 1-form gauge theories.
- Everytime we go up in dimension, one side goes from p-form to $(p+1)$-form.
- In general, in D dimension, we have an equivalence betweewn p-form and $(d-p-2)$-form $U(1)$ gauge theories.

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- If we complete the square and integrate out B, we recover the original YMs action for A.
- However, if we integrate out A, we get the constraint $dB = 0$. Therefore we can write $B = d\tilde{A}$, for a $(d-p-2)$-form. We ge the YMs action for $\tilde{A}$.

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- $\hat{A}$ here is the Pontryagin dual group $Hom(A, U(1))$. Finite abelian groups are "self-dual": $\widehat{\mathbb{Z}_N} \simeq \mathbb{Z}_N$. This also explains why there is a dimension shift between $U(1)$ and finite groups, because the dual of $U(1) = Hom(U(1), U(1)) = \mathbb{Z}$.

Examples in low dimensions: $0 + 1$ D

- In $0+1$D, aka quantum mechanics. Sigma model to A means the Hilbert space is $\mathbb{C}[A]$, one state for each element $a \in A$.

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- This is the discrete analogue of Fourier transform, where $\chi(x, p) = e^{ipx}$ and

$$\delta_x \mapsto \int_p e^{ipx} \delta_p = e^{ipx}$$

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- The abelian duality states that they are equivalence of fusion categories. This is precisely the character theory for finite abelian groups: irreducible representations of A are all one dimensional, labelled by a character $\alpha \in \hat{A} : A \to U(1)$.

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- When A is abelian, we have the $Z(\mathrm{Rep}_A) = \mathrm{Rep}_A \times \mathrm{Rep}_{\hat{A}}$. This is an abelian topological order where each anyon is labelled by its A-charge $\in \mathrm{Rep}_A$ and its S matrix pairing, which defines a map $\hat{A} \to U(1)$, that is a $\hat{A}$ representation. We see that $Z(\mathrm{Rep}_A) = Z(\mathrm{Rep}_{\hat{A}})$.

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- Take $A = \mathbb{Z}_2$, this is the toric code topological order. Then this is the electro-magnetic duality that exchanges e and m anyons.
- Remark: currently there is a lot of work on symmetry TFT. In the end, I will relate this duality to gauging/ungauging, Kramers-Wannier duality via symmetry TFTs.

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- Lastly, let Vect be the category of complex vector spaces and linear maps.

TFTs and symmetric monoidal categories

- A topological field theory is a symmetric monoidal functor $Z : \mathrm{Bord}_D \to \mathrm{Vect}$. It assigns a vector space $Z(N)$ to each $D-1$ manifold N, and a linear map $Z(M) : Z(N) \to Z(N')$ for a bordism $M : N \to N'$.

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- For every bordism $M : N \to N'$, we have a commutative diagram:

$$\begin{array}{ccc} Z_1(N) & \xrightarrow{Z_1(M)} & Z_1(N') \\ \downarrow F(N) & & \downarrow F(N) \\ Z_2(N) & \xrightarrow{Z_2(M)} & Z_2(N') \end{array}$$

Finite homotopy TFTs

- Now let's define these p-form gauge theories. Fix an abelian group A, a form level p and a dimension D. We are going to define $Z_{A,p} : \mathrm{Bord}^{or} \to \mathrm{Vect}$. This is also called untwisted Dijkgraaf Witten theory.

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- Sanity check: in $2+1$D, take $A = \mathbb{Z}_2$, $p = 1$, and $N = T^2 = S^1 \times S^1$. This theory $Z_{A,p}$ corresponds to the toric code. We see $Z_{A,p}(N) = \mathbb{C}[H^1(T^2, \mathbb{Z}_2)] = \mathbb{C}^4$, which is the dimension of anyons in toric code.

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$$\mathbb{C}[H^p(M;A)] \xrightarrow{Z_{A,p}(M)} \mathbb{C}[H^p(M;A)]$$

$$a \longmapsto C \sum_{q(b)=a} p(b).$$

for $a \in H^p(N;A)$, viewed as an basis element of $\mathbb{C}[H^p(N;A)]$. Similarly, $b \in H^p(M;A)$.

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- C is some constant which is needed for composition of bordisms.

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Fix dimension D and form level p. Let A be a finite abelian group. Then we have an equivalence of higher form finite gauge theories, as functors from $\mathrm{Bord}^{\mathsf{or}} \to \mathrm{Vect}$:

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- In fact, we have a generalization of this statement for abelian higher groups symmetry, which goes under the name of π-*finite spectra*.

Orientation and Poincare duality

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Theorem (Poincare duality)

Let N be an oriented $D-1$ dimensional manifold, let $[N] \in H^{D-1}(N;\mathbb{Z})$ be foundamental class. then we have an equivalence of groups:

$$\int_N : H^p(N;A) \simeq H_{D-1-p}(N;A)$$

where the right hand side is the $(D-1-p)$-th homology groups.

Pontryagin-Brown-Comenetz duality

- Recall that the Pontryagin dual $\hat{A}$ of abelian group A is defined as $Hom(A, U(1))$. This forms a duality for finite abelian groups: $\widehat{\hat{A}} = A$. There is a non-denegerate bilinear pairing $\chi : A \otimes \hat{A} \to U(1)$.

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- Let N be a topological space, then we have an induced pairing on cohomology/homology groups:

$$H_p(N; A) \otimes H^p(N; \hat{A}) \to H_0(N; A \otimes \hat{A}) \xrightarrow{\chi} U(1).$$

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- Brown-Comenetz duality shows that this is in fact a non-degenerate pairing:

Theorem (Pontryagin-Brown-Comenetz duality)

The pairing above makes $H^p(N; \hat{A})$ the Pontryagin dual of $H_p(N; A)$.

Isomorphism on state spaces

Reminder:

$$\int_{[N]} : H^p(N;A) \simeq H_{D-1-p}(N;A), \widehat{H_p(N;A)} = H^p(N;\hat{A}).$$

- With these 2 theorems in hand in hand, we will construct the isomorphism $Z_{A,p} \simeq Z_{\hat{A},D-p-1}$ on state space. Given a d dimensional closed manifold, we want to give an isomorphism

$$F(N) : \mathbb{C}[H^p(N;A)] \simeq \mathbb{C}[H^{D-p-1}(N;\hat{A})].$$

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- Poincare duality and Brown-Comenetz duality implies that they are Pontryagin dual:

$$\widehat{H^p(N;A)} = \widehat{H_{D-1-p}(N;A)} = H^{D-p-1}(N;\hat{A})$$

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$$F(N) : \mathbb{C}[H^p(N;A)] \simeq \mathbb{C}[H^{D-p-1}(N;\hat{A})].$$

- Poincare duality and Brown-Comenetz duality implies that they are Pontryagin dual:

$$\widehat{H^p(N;A)} = \widehat{H_{D-1-p}(N;A)} = H^{D-p-1}(N;\hat{A})$$

- Therefore discrete Fourier transform gives an isomorphism

$$\mathbb{C}[H^p(N;A)] \to \mathbb{C}[H^{D-p-1}(N;\hat{A})]$$

$$a \mapsto \sum_{\alpha} \alpha(\int_{[N]} a) \cdot \alpha$$

Application to symmetry TFTs

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- As $D-1$ dimensional theory Z with $(p-1)$-form A symmetry can be viewed as boundaries of the D dimensional $Z_{A,p}$ TFT, we see that we have an extension of gauging/ungauging to general dimensions, where the dual theory has $(D-p-1)$-form $\hat{A}$ symmetries.

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- Over other coefficients: chromatic abelian duality.