Smith Homomorphisms and Anomaly Matching

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Objectives

- Use Smith homomorphisms to match anomalies in the symmetry broken phase, generalizing the $\mathbb{Z}_2$ case done in [1, 2].
- Find an anomaly obstruction to having a spontaneously broken fully-gapped phase, where the symmetry is broken by some order parameter.
- Perform the anomaly matching procedure in free-fermion theories.

Introduction

A d dimensional field theory Z with global symmetry G can have 't Hooft anomalies. By [3], these are classifed by cobordism invariants $\Omega^{d+1}(BG)$. 't Hooft anomalies are powerful invariants, as they are preserved under any deformation, including RG flow.

Following [1], we investigate anomaly matching/constraints in the symmetry broken phase, where some symmetry of G is broken in the IR. The symmetry is broken by an order parameter that transforms in some representation V of G. In this phase, there are domain walls/defects when we set twisted boundary conditions. There are confined degrees of freedom that live over these defects. They have (twisted) G symmetry with anomalies. Following [1], we perform anomaly matching on these defects via Smith homomorphisms.

Smith Homomorphism

Smith homomorphisms are maps between different bordism groups of different dimensions, and thus anomalies of different dimensions. Given

- a homotopy type X
- a tangential structure $\eta : X \to BO$
- an n-dimensional vector bundle $\rho : X \to BO_n$

there is a fiber sequence of Thom spectra:

$$S_X(\rho)^\eta \to X^\eta \xrightarrow{sm} \Sigma^{-n} X^{\eta+\rho},$$

where $S_X(\rho)$ is the sphere bundle of ρ over X.

Anomaly Matching Map

Following [3], the anomalies of a d dimensional field theory with tangential structure $\eta : X \to BO$ are classified by $\Omega^{d+1}(X^\eta)$.

Taking Ω^*, we get a long exact sequence of anomaly groups:

$$\cdots \to \Omega^{*-n}(X^{\eta+\rho}) \xrightarrow{sm} \Omega^*(X^\eta) \to \Omega^*(S_X(\rho)^\eta) \to \cdots$$

Periodic Families

Often as we iterate the Smith homomorphism, the Thom spectrum X^η repeats. That is,

$$X^{\eta+k\rho} \simeq \Sigma^{kn} X^\eta.$$

Here are some examples of periodic families:

X	ρ	Period	Symmetry
$B\mathrm{Spin} \times B\mathbb{Z}_2$	sign rep	4	$\mathrm{Spin} \times \mathbb{Z}_2$ Pin^- $\mathrm{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4$ Pin^+
$B\mathrm{Spin} \times BU(1)$	charge 1	2	$\mathrm{Spin} \times U(1)$ Spin^c
$B\mathrm{Spin} \times BSU(2)$	2	1	$\mathrm{Spin} \times SU(2)$
$B(\mathrm{Spin} \times_{\mathbb{Z}_2} SU(2))$	3 real	2	$\mathrm{Spin} \times_{\mathbb{Z}_2} SU(2)$ $\mathrm{Spin} \times SO(3)$

Hypothesis

Given

- a tangential structure $\eta : X \to BO$.
- a d dimensional field theory Z with tangential structure X and anomaly $\alpha \in \Omega^{d+1}(X^\eta)$.
- a symmetry-breaking order parameter ϕ transforming in $\rho : X \to BO_n$.
- the IR is gapped,

then

- the defect created by twisted boundary condition has excitation localized at $\langle\phi\rangle = 0$.
- It has anomaly $\beta \in \Omega^{d+1-n}(X^{\eta+\rho})$ such that

$$sm : \beta \mapsto \alpha.$$

Free Fermion Symmetry Breaking

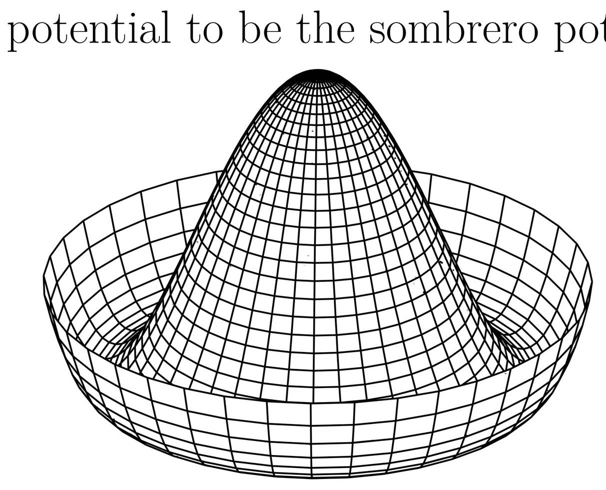
Consider a Dirac fermion $\psi = (\psi_L, \psi_R)$ in 3+1d with $U(1)$ symmetry and charges $(n+1, n)$. With order parameter a charge -1 boson ϕ, the Yukawa coupling

$$\phi \bar{\psi}_R \psi_L + h.c.$$

is $U(1)$-invariant. Consider the Lagrangian:

$$\mathcal{L} = i\bar{\psi}\partial_\mu \gamma^\mu \psi + \phi\bar{\psi}_R\psi_L + h.c. + \partial_\mu \phi^* \partial^\mu \phi + V(|\phi|^2)$$

We set the potential to be the sombrero potential [4]:



$$V(|\phi|^2) = -\frac{m^2}{2}|\phi|^2 + \frac{\lambda}{4!}|\phi|^4,$$

As ϕ condenses, the $U(1)$ symmetry is broken.

Twisted Boundary Condition

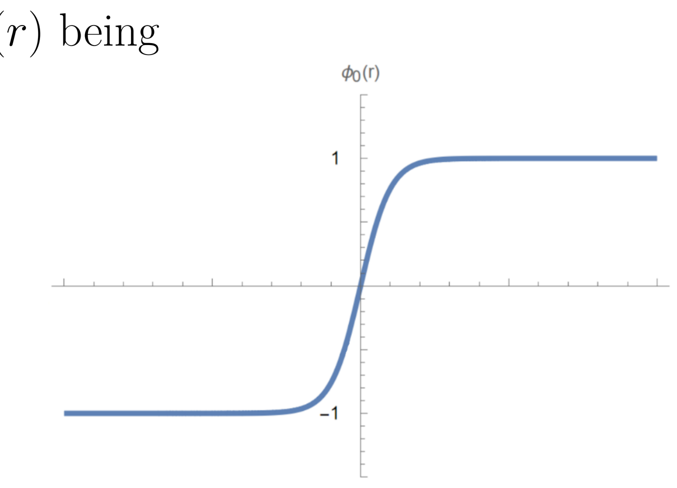
Using polar coordinates

$$(x^2, x^3) = (r\cos\theta, r\sin\theta),$$

we set twisted boundary conditions:

$$\phi(r,\theta) = \phi_0(r)e^{i\theta}$$

with $\phi_0(r)$ being



Solving EOM

Take the ansatz $\psi = h(r)\Psi(x^0, x^1)$. The EOM becomes

$$i\,\partial_i h(x^i)\,\gamma^i_L\,\psi_L = \phi_0(r)\,e^{-i\theta}\,h(x^i)\,\psi_R \qquad (0.1)$$
$$i\,\partial_i h(x^i)\,\gamma^i_R\,\psi_R = \phi_0(r)\,e^{+i\theta}\,h(x^i)\,\psi_L. \qquad (0.2)$$

Solving this, we get

$$\gamma^2\Psi = \begin{pmatrix}\gamma^2_R\,\psi_R \\ \gamma^2_L\,\psi_L\end{pmatrix} = \begin{pmatrix}-\psi_L \\ -\psi_R\end{pmatrix}$$
$$\gamma^3\Psi = \begin{pmatrix}\gamma^3_R\,\psi_R \\ \gamma^3_L\,\psi_L\end{pmatrix} = \begin{pmatrix}-i\psi_L \\ i\psi_R\end{pmatrix}.$$

Therefore Ψ is a left-handed chiral fermion in $1+1$d.

$\mathbb{Z}_2$ case

In the $\mathbf{Z}_2$ case, the tangential structures are 4-periodic, connected by Smith homomorphisms:

$$\mathrm{Spin} \times \mathbb{Z}_2 \rightsquigarrow \mathrm{Pin}^- \rightsquigarrow \mathrm{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \rightsquigarrow \mathrm{Pin}^+.$$

Here are the anomaly classes in different dimensions:

d+1	Spin	$\mathrm{Spin} \times \mathbb{Z}_2$	Pin^-	$\mathrm{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4$	Pin^+
0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	0
2	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_8$	0	$\mathbb{Z}_2$
3	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_8$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
4	0	0	0	0	$\mathbb{Z}_{16}$
5	0	0	0	$\mathbb{Z}_{16}$	0
6	0	0	$\mathbb{Z}_{16}$	0	0

$U(1)$ Case

We focus on the $3+1$ d to $1+1$d. For $U(1)$ symmetry, there is a 2-periodic spin family:

$$\mathrm{Spin} \times U(1) \rightsquigarrow \mathrm{Spin}^c = \mathrm{Spin} \times_{\mathbb{Z}_2} U(1)$$

As mentioned, there is a massless 1+1d left-handed chiral fermion Ψ living at $x^2 = x^3 = 0$. The charge of Ψ is

$$(n + \frac{1}{2}).$$

The fractional charge means that it has Spin^c symmetry. Both $3+1$ d fermionic $U(1)$ theories and $1+1$ d Spin^c theories have 2 perturbative anomalies:

Symmetry	Dimension	Anomaly Polynomials
$\mathrm{Spin} \times U(1)$	$3+1$	$tr(\gamma^5 c)$ $tr(\gamma^5 c^3)$
Spin^c	$1+1$	$tr(\gamma^3)$ $tr(\gamma^3 c^2)$

The anomaly matching becomes:

$$sm : (1, (n+\frac{1}{2})^2) \mapsto (n+1-n, (n+1)^3 - n^3)$$

which can be realized by a linear map

$$\begin{pmatrix} 1 & 0 \\ 1/4 & 3 \end{pmatrix}$$

We can generalize this to perturbative chiral $U(1)$ anomalies in general (even) dimensions.

Future Work

- For continuous symmetry, relate the anomaly matching on the defect with the WZW term for the IR sigma model to the vacua.
- Extend the anomaly matching to higher form symmetries and higher groups, and beyond free-fermions.

References

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