BRAIDED MONOIDAL 2-CATEGORIES, KNOT HOMOLOGY, AND 3+1D TQFTS

YU LEON LIU

This is work in progress with Aaron Mazel-Gee, David Reutter, Catherina Stroppel, and Paul Weldrich.

1. Motivation

One of the most exciting part of mathematical physics is the interplay between three seeming different objects:

$$\begin{tikzcd}
& \text{Braided monoidal Categories (UMTC)} \arrow[dl, leftrightarrow, "RT"] \arrow[dr, leftrightarrow] & \\
2+1\text{ D TQFT} \arrow[rr, leftrightarrow, "Witten"] & & \text{knot polynomial}
\end{tikzcd} \tag{1.1}$$

A particular example is

$$\begin{tikzcd}
& U_q(sl_2) \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\text{Chern-Simons theory} \arrow[rr, leftrightarrow] & & \text{Jones polynomial}
\end{tikzcd} \tag{1.2}$$

Remark 1.3. Let's recall what a braided monoidal structure is on a category C. First a monoidal structure is an associative tensor product $\otimes$ on C. A braided monoidal structure on $(C, \otimes)$ is a braiding $\beta : x \otimes y \simeq y \otimes x$ that satisfies the Hexagon axioms, one of them relates braiding x over $(y \otimes z)$ and first braid x over y and then braid it over z. The other one is about braiding z under x and y.

In the 2000's, Khovanov and collaborators categorified knot polynomials to knot homologies theories. That is, instead of attaching a polynomial $J(L)$ to a link L (really a link diagram), they attached a chain complex to each link diagram and recovers knot polynomial by taking alternating sum of dimensions. Note that in this case, chain complexes attached to different link diagrams of the same link are homotopy equivalent to each other, this is some coherence structure we need to take in account.

This raises a natural question:

Question 1.4. Is there an analogues picture replacing knot polynomials with knot homologies?

We also need to categorify the other two sides, there are natural answers:

$$\begin{tikzcd}
& \text{Braided monoidal 2-Categories} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
3+1\text{ D TQFT} \arrow[rr, leftrightarrow] & & \text{knot homologies}
\end{tikzcd} \tag{1.5}$$

In this talk, we discuss partial progress in this answer, namely connecting knot homologies and braided monoidal 2-categories.

Remark 1.6. Here's some of the difficulties:

(1) Since we care about chain complexes up to homotopy, they are really $(\infty, 2)$ categories, aka they have invertible morphisms of all levels. They encode data like surface corbordism of links and braids.
(2) The right notion of braided monoidal for ∞ categories is $\mathbb{E}_2$ algebra by May. Heauristically, it is encoding fusion of points in $\mathbb{R}_2$.
(3) While $\mathbb{E}_2$ algebra is homotopically a natural notion, its difficult to give a generator and relation definition. Our 2 category is going to be really concrete and not formal, thus it is difficult to define a define a $\mathbb{E}_2$ structure

For the rest of the talk, I will sketch a construction of a $\mathbb{E}_2$ $(\infty, 2)$ category, and give a receipt for how to do this in general.

2. The monoidal $(2, 2)$ category of Soergel bimodules

First we will give a construction of the monoidal $(\mathbb{E}_1)$ $(2, 2)$ category of Soergel bimodules. This is baiscally a review of Soergel bimodule.

Definition 2.1. Let k be a field of characteristic 0. We denote the graded polynomial free on n generator $k[x_1, \cdots, x_n]$ as R_n, with degree $|x_i| = 2$. Furthermore, let $R_n - \mathrm{BMod}$ be the abelian category of (graded) R_n bimodules. It is monoidal under $\otimes_R$. The grading will be present but not important for this talk, so I will drop it.

Now we will define the monoidal 1 category of Sbim_n:

Construction 2.2. Given $s = (i, i+1) \in S_n$, which acts on R_n by permutting the generators. Let R_n^s be the subalgebra of invariance. For $n = 2$, $s = (12)$, then $R_2^s = k[x_1 + x_2, x_1 \cdot x_2]$. Let B_s denote the $R_n - R_n$ bimodule

$$R_n \otimes_{R_n^s} R_n. \tag{2.3}$$

Let Sbim_n be the smallest Karoubi-complete full monoidal subcategory of $R_n - \mathrm{BMod}$ that contains R and B_s.

Remark 2.4. Objects in Sbim_n are idempotents of tensors under R of B_{s_i} for different s_i.

Remark 2.5. Sbim_n is an additive category, but it is not abelian. More specifically, it doesn't have kernels and cokernels. Crucially, the symmetric bimodule $R_n(s)$, where as a $R_n - R_n$ the left R_n is twisted, is not a Soergel bimodule. Note that it implements the S_n action on $R_n - \mathrm{BMod}$. Therefore there is not a S_n action on Sbim_n, therefore we have room for a braiding.

Aside from the monoidal product on Sbim_n, there is also a product that goes between them:

Construction 2.6. There is an exterior product $\boxtimes = \otimes_k : \mathrm{Sbim}_n \times \mathrm{Sbim}_m \to \mathrm{Sbim}_{n+m}$, coming from the natural map $R_n \otimes_k R_m \simeq R_{n+m}$ by stacking the generators.

We can construct a monoidal 2-category $\underline{\mathbb{S}\mathrm{bim}}$ that captures all of the data above:

Definition 2.7. Let $\underline{\mathbb{S}\mathrm{bim}}$ be the $(2, 2)$ category with objects $n \in \mathbb{N}$, and morphisms

$$Hom_{\underline{\mathbb{S}\mathrm{bim}}}(n, m) = \begin{cases} \mathrm{Sbim}_n & n = m \\ 0 & n \neq m \end{cases} \tag{2.8}$$

Note that being the endormorphism category, it uses the monoidal structure of Sbim_n.

Remark 2.9. We can add in more morphism then just from n to n. We will discuss this at the end of the talk.

Moreoever, the $\boxtimes$ makes $\underline{\mathbb{S}\mathrm{bim}}$ into a monoidal $(2, 2)$ category. Now we move on to building a braided monoidal structure. However, $\underline{\mathbb{S}\mathrm{bim}}$ doesn't have a braided monoidal structure. This is because the braiding is implemented using chain complexes of Soergel bimodules. Therefore we need to discuss that first.

3. Braid group action on $K^b\mathrm{Sbim}_n$

Now let's recall the classical story of Ronquier's braid group Br_n actions on Sbim_n. Before we start, let me just remark that there are a lot of different ways to describe this, one reason is by thinking about category $\mathcal{O}$ for gl_n. Then the reason why the n-th braid group Br_n acts on it, rather than some other group, is because Br_n is the braid group associated to the Coexeter group S_n of gl_n.

Since Br_n is generated by the same coexeter generators $s_i = (i, i+1)$, let's just describe its action. Since Sbim_n is monoidal, we can just find a bimodule to implement this. However, it turns out that we need to go to chain complex $K^b\mathrm{Sbim}$:

Definition 3.1. $K^b\mathrm{Sbim}_n$ is the $(\infty, 1)$ category of bounded chain complex of Soergel bimodules, localized at homotopy equivalence. It is monoidal as $K : \mathrm{AddCat} \to \mathrm{StableCat}$ from additive $(\infty, 1)$ category to stable $(\infty, 1)$ categories is symmetric monoidal [**?**].

Remark 3.2. Given an additive category $\mathcal{C}$, $K^b\mathcal{C}$ as an $(\infty, 1)$ category, satisfies an extremely useful universal property, which is that it is the universal stable category associated to $\mathcal{C}$. That is, any additive functor $\mathcal{C} \to \mathcal{D}$ with $\mathcal{D}$ stable is equivalent to an exact functor from $K^b\mathcal{C} \to \mathcal{D}$. This universal property is crucial in the construction of the braided monoidal $(\infty, 2)$ category. However, it is only available when we work with the ∞ context, and not present for its homotopy category $h_1(K^b\mathcal{C})$. In this sense, the fully coherent ∞ version of the proof is simpler than the 2-category version.

Remark 3.3. Note that Sbim_n is only an additive category, not abelian. Therefore we cannot define its derived category.

Construction 3.4. Given s_i, we assign the chain complex $F(s_i)$:

$$F(s_i) = \cdots \to 0 \to \underline{R_n \otimes_{R_n^s} R_n} \to 0 \to \cdots \tag{3.5}$$

where $\underline{\cdots}$ signals the degree 0 part.

Ronquier showed the following:

Theorem 3.6. *This assignment defines a fully coherent monoidal map from $Br_n \to K^b\mathrm{Sbim}_n$.*

Remark 3.7. The category of $R_n - \mathrm{BMod}$ has a S_n action by permuting the coordinates. This is implemented by the permutation bimodules$R_n(s)$, which are R_n as their underlying vectors, and their right actions are standard one. However their left actions are twisted. For the generator $s_i = (i, i+1)$, the left action on $R_n(s)$ swaps the x_i and x_{i+1} generators. Tensoring by these permutation bimodules implements the S_n twist.

However, crucially, as mentioned above, these bimodules are not Soergel bimodules, that is, they don't live in Sbim_n or $K^b\mathrm{Sbim}$.

Proof. Let us give the central idea of the proof: the idea is that these Ronquier complexes $F(s)$ satisfies a very special property: there is a natural functor from $K^b\mathrm{Sbim}_n \to D^b(R_n - \mathrm{BMod})$, by viewing a chain complex of Soergel bimodules in the derived category of R_n bimodules. In $D^b(R_n - \mathrm{BMod})$, the chain complex $F(s)$ is quasi-coherent to the permutation bimodule $R_n(s)$ described above. Therefore this action recovers the symmetric group action on $D^b(R_n - \mathrm{BMod})$.

From there, Ronquier lifted the coherence of the symmetric group action on $D^b(R_n - \mathrm{BMod})$ to the coherence data of the braid group action on $K^b\mathrm{Sbim}_n$. In some sense, the braid group ation on $K^b\mathrm{Sbim}_n$ is deformed from the symmetric group action on $D^b(R_n - \mathrm{BMod})$. Diagrammatically, we have the following commutating diagram:

$$
\begin{tikzcd}
BBr_n \arrow[r] \arrow[d] & K^b\mathrm{Sbim}_n \arrow[d] \\
BS_n \arrow[r] & D^b(R_n - \mathrm{BMod})
\end{tikzcd}
\tag{3.8}
$$

□

4. Braided monoidal $(\infty, 2)$ categories.

This gives a hint of how to build the braided monoidal 2-categories, namely we can use the $(\infty, 2)$ $K^b\underline{\mathbb{S}\mathrm{bim}}$ cateogory build with the n-th endormorphism category being the $(\infty, 1)$ category $K^b\mathrm{Sbim}_n$:

Definition 4.1. Let $K^b\underline{\mathbb{S}\mathrm{bim}}$ be the $(\infty, 2)$ category with objects being $n \in \mathbb{N}$, and the $(\infty, 1)$ category of morphism being:

$$
\begin{cases} K^b\mathrm{Sbim}_n & n = m \\ 0 & n \neq m. \end{cases}
\tag{4.2}
$$

Note that this uses the monoidal structure of $K^b\mathrm{Sbim}_n$ as the internal composition of endomorphisms. It has a monoidal $(\mathbb{E}_1)$ structure coming from the exterior product.

We had maps of $(\infty, 1)$ category from $K^b\mathrm{Sbim}_n \to D^b(R_n - \mathrm{BMod})$. We want a similar maps after packaging $K^b\mathrm{Sbim}_n$ to the $(\infty, 2)$ category $K^b\underline{\mathbb{S}\mathrm{bim}}$. The way to do that is by realizing that $D^b(R_n - \mathrm{BMod})$ are exactly the endomorphism category of R_n in the $(\infty, 2)$ category of derived Morita category Mor(dgVect).

Definition 4.3. Let Mor(dgVect) be the $(\infty, 2)$ category whose objects are algebras and morphisms are (derived) bimodules. It has a symmetric monoidal $(\mathbb{E}_\infty)$ structure by $\otimes_k$.

The construction above defines a monoidal $K^b\underline{\mathbb{S}\mathrm{bim}} \to \mathrm{Mor(dgVect)}$.

Now we can state the final theorem:

Theorem 4.4 (Mazel-Gee, L.L., Reutter, Stroppel, Wedrich)**.** *There is a $\mathbb{E}_2$ structure on* K^bSbim *such that the braid group action on the n-th object is precisely by Ronquier complexes $F(s)$, making this diagram commute:*

$$
\begin{tikzcd}
\mathbb{E}_1 \arrow[r] \arrow[d] & \underline{\mathbb{S}\mathrm{bim}} \arrow[d] \\
\mathbb{E}_2 \arrow[r, red] \arrow[d] & K^b\underline{\mathbb{S}\mathrm{bim}} \arrow[d] \\
\mathbb{E}_\infty \arrow[r] & \mathrm{Mor}(dgVect)
\end{tikzcd}
\tag{4.5}
$$

In words, this $\mathbb{E}_2$ structure on $K^b\underline{\mathbb{S}\mathrm{bim}}$'s underlying monoidal structure is the one inherited from $\underline{\mathbb{S}\mathrm{bim}}$, and the map $K^b\underline{\mathbb{S}\mathrm{bim}} \to \mathrm{Mor(dgVect)}$ takes the $\mathbb{E}_2$ structure on $K^b\underline{\mathbb{S}\mathrm{bim}}$, namely its braiding, to the symmetric braiding on Mor(dgVect). On the level of braid group action on the n-th object, this is precisely the diagram Equation (3.8) above.

Remark 4.6. To proof this theorem, just like how quantum group reps are deformed from representations of sl_n, which is symmetric monoidal, we also deform from a symmetric monoidal structure. More specifically, an $\mathbb{E}_2$ algebra structure on a $(\infty, 2)$ category requires infinite coherence. The main technical result is to lift the higher cells data from the $\mathbb{E}_\infty$ aka symmetric monoidal structure.

5. Applications/future directions

Let me just mention some applications/future directions here:

(1) Our main machinery is a mechanism that can produce $(\infty, 2)$ categories from concrete data. Using this we can produce many other braided monoidal $(\infty, 2)$ categories, many which are much more relevant in the triangle above.
(2) $K^b\underline{\mathbb{S}\mathrm{bim}}$ currently is a pretty lame category, it has no non-trivial non-endormophisms. We can easily add this non-endormorphism, going by the name of singular Soergel bimodules.
(3) $K^b\underline{\mathbb{S}\mathrm{bim}}$ is not dualizable, by the simple reason that the polynomial rings R_n are not finite dimensional. The triply graded homology is also should be thought of as sl_∞ homology. We have work in progress on how to create fully dualizable braided monoidal $(\infty, 2)$ categories, the ones that can be plugged into TQFTs and get manifold invariants.

1 Oxford St, Cambridge, MA 02139
Email address: `yuleonliu@math.harvard.edu`