STRATIFICATION THEORY IN QUANTUM PHYSICS

YU LEON LIU

This is notes for my talk for the UCLA algebraic topology 2022 seminar.

1. Physical Motivation

Physical systems often comes in families, where we can tune knobs such as temperature, pressure, and the coupling constants. That is, we would like to study families of physical theories over the space of coupling constants. Interestingly, much of the low-energy, long-distance dynamics are indepedent are invariant (at least vary continuously) under small deformation of the constant, forming some sort of local systems. However, there are lower dimensional critical submanifolds where interesting behaviors occur, such as the presence of ground state degeneracy, or critical phenonemon. To study these, physicists have drawn out phase diagrams, which cuts the parameter spaces into different pieces, or stratums. Mathematically, we call these spaces cut into different pieces stratified spaces. Furthermore, there is a notion of constructible sheaves on them, which are local systems on each stratum, but the dimensions can jump across stratums.

The study of stratified spaces and constructible sheaves on them is everywhere in mathematics, from singularity theory to geometric representation theory to genuine equivariant homotopy theory [**?**]. A new homotopic approach, called exit-paths categories [**?**], has been successful in understanding the structure of constructible sheaves. In particular, it says

(1) The homotopical data of a stratified space X is the homotopy type of each stratm X_p, the space of exit paths (links) $L(p, q)$ between each two stratum, the space of higher exit paths that forms composition... They are packaged together into a nice ∞ category called the exit path (∞) category $\mathcal{E}xit(X)$ associated to X.
(2) The data of a constructible sheaf $\mathcal{F}$ can be phrase in terms of $\mathcal{E}xit(X)$: a local system $\mathcal{F}_{|X_p}$ on each stratum X_p, and a map of local systems pulled back over the link $L(p, q)$ for each pair p, q, and higher compatibilities...

In this talk we will first review the theory of exit-paths approach to stratification, then apply it to physical systems and see how interesting math (and physics!) pops out almost immediately. To this end we will give two examples:

(1) Berry phase and Borel-Weil: we will construct a natural G-equivariant quantum mechanical system which recovers Borel-Weil and some parts of Beilinson-Bernstein.
(2) (Spherical) symmetry broken phases in QFT: then we will study anomaly matching in spontaneous symmetry breaking phases, and see Thom isomorphism and Smith homomorphism naturally appears.

First is joint work with Ryan Thorngren, second is joint with Arun Debray, Sanath Devalapurkar, Cameron Krulewski, Natalia Pacheco-Tallaj, and Ryan Thorngren.

2. Stratification and local systems

Definition 2.1. A stratified space is a space X together with a continuous map $X \to P$, where P is a poset, which we viewed as a topological space by the Alexander topology. TODO[inline]define Alexander topology....

Heauristically, this means that

(1) for each $p \in P$, we have a locally closed subspace $X_p \subset X$. Its closer is denoted as $\bar{X}_p$. As a set, $X = \sqcup_p X_p$.
(2) For $p < q$, then $X_p \subset \bar{X}_q$. We will only consider nice examples, where ...

Example 2.2 (Stratification of vector spaces)**.** Here's a simple example that will show up later in this talk, let $X = V$ be a vector space, and $P = [1] = (0 \to 1)$. The fibers over 0 and 1 are the origin 0 and $\mathbb{R} - \{0\}$. When we have a group G and V is a G-representation, this will be a G-equivariant stratification. As we will see later, this example is a good linear model for understanding stratification theory, and is related to Thom isomorphism and spontaneous symmetry breaking.

Example 2.3 (Coadjoint orbits)**.** Here's a more sophisticated example: let $\mathfrak{g}^*$ be the coadjoint representation of a simple compact lie group G [1]. The stabilizer of each point $v \in \mathfrak{g}^*$ is a parabolic subgroup $P \subset G$. Fix a maximal torus $T \subset G$, every parabolic is conjugate to a standard parabolic P, that is, $T \subset P$. The standard parabolics forms a poset, and $\mathfrak{g}^*$ is stratified over that poset. For $SU(2)$, $\mathfrak{su}_2 \simeq \mathbb{R}^3$. Rotation around z axis defines maximal Cartan $U(1)^z$. There are only two standard parabolics $U(1)^z \subset SU(2)$. So the poset is [2] with fibers 0 and $\mathbb{R}^3 - 0$.

Remark 2.4. Stratified spaces show up all over mathematics. Here's some examples:

(1) Singular varieties are stratified by their singularities.
(2) In general, A G-space X is stratified by the (conjugacy) class of the stabilizers. One important example in geometric representation theory is the Bruhat stratification on the flag variety $\mathcal{B} = G/B$, stratified over the Borel group B acting on G/B on the left.

Given a stratified space X, there is a natural notion of constructible sheaf. Let's recall some sheaf theory:

Definition 2.5. Let X be a space and a category C, then we have the site of open subsets $Op(X)$. The morphisms are inclusions and open covers are open covers. A (C)-valued sheaf is a functor $Op(X)^{op} \to C$ that satisfies descent.

Typical choices of C are sets, vector spaces, and more ∞-categorically, spaces and spectra. Soon we will take another leap of faith and consider C to be the category of QFTs. We will suppressed C when it is clear.

Before defining constructible sheaf, we first define the notion of a local system:

Definition 2.6. A sheaf $\mathcal{F}$ on X is a local system, if for all $x \in X$, there is a neighborhood U containing x such that $\mathcal{F}_{|U}$ is a constant sheaf. Let $\mathrm{LocSys}_C(X)$ be the category of local systems on X.

Now we can explain what a constructible sheaf is:

Definition 2.7. A sheaf $\mathcal{F}$ on a stratified space $X \to P$ is a constructible sheaf if for all stratum X_p, $\mathcal{F}_{|X_p}$ is a local system on X_p. Let $\mathrm{Constr}_C(X)$ be the category of constructible sheaves.

Example 2.8 (Gauss-Mannin connection)**.** Consider a smooth morphism $X \to B$ of characteristic 0 varieties, then the fiberwise cohomology groups $H^*_{dR}(X_b)$ forms a local system over B. When there are singularities, we can stratify B by the singularities of the fibers, and the fiberwise cohomology is no longer a local system but rather a constructible sheaves.

3. Homotopy theory of constructible sheaves and exit path categories

How can we understand constructible sheaf homotopically? First we turn to local systems.

[1] Note we are using the real compact versions rather than the complex version, though they are equivalent.

3.1. **Local systesm and ∞-groupoid $\pi_\infty X$.** There is path-approach to understanding local systems.

Definition 3.1. Given a space X, we can define the ∞-groupoid $\pi_\infty X$ whose objects are points in X, morphisms are paths, 2-morphisms are homotopy between paths...

This ∞-groupoid remembers all the paths information of X.

Fix a local system $\mathcal{F}$. Given $x \in X$, we can assign the stalk $\mathcal{F}_x$. Furthermore given a path $p : [0,1] \to X$ from $p(0) = x$ to $p(1) = y$, then this determines an isomorphism $p : \mathcal{F}_x \simeq \mathcal{F}_y$. Furthermore, any homotopy between two such paths will define a homotopy between such two isomorphisms. In fact, this construction defines a functor:

$$mon : \mathrm{LocSys}_C(X) \to Fun(\pi_\infty X, C) \tag{3.2}$$

In fact, for X nice enough, this is an equivalence:

Theorem 3.3 ([?])**.** *mon is an equivalence of categories.*

Remark 3.4. Since $\pi_\infty X$ only depends on the homotopy type of X, we see that $\mathrm{LocSys}(-)$ is a homotopy invariant.

This gives a homotopic approach to understand local systems, it is simply a functor from the ∞-groupoid $\pi_\infty X$ to our target category C.

Note that $Fun(\pi_\infty X, C)$ is a path-theoretic definition of local systems, as oppose to the sheaf definition which is focused more on open sets.

3.2. **Exit path categories.** To generalize this to constructible sheaves, we need to find the analogue for $\pi_\infty X$. Recall that $\pi_\infty X$ encodes the path (and higher path) data of X. In the case that X has more than one stratum, we need to be more selective about what kind of paths are allowed.

Consider an open closed decomposition, where $X = X_0 \sqcup (X - X_0)$, where X_0 is a closed subset. Take $x \in X_0$ and $y \in X - X_0$, given a path $p : [0,1] \to X$ from x to y such that $x = p(0) \in X_0$ and $p((0,1]) \subset X - X_0$, then we see that we can only transport $\mathcal{F}$ in one direction:

Any element of $\mathcal{F}_x$ can be lifted to an open set, thus we can restrict to some element of $\mathcal{F}_t$ for $t \in (0,1]$. Now since $p((0,1])$ lives entirely in $X - X_0$ and $\mathcal{F}_{X-X_0}$ is a local system by definition, we see that we can transport it to y. Therefore we get a map $\mathcal{F}_x \to \mathcal{F}_y$, but crucially in reverse. Note that this map also doesn't have to be invertible.

Definition 3.5. Given a stratified space $X \to P$, and two points $p < q \in P$. An exit path in X over p, q is a map $p : [0,1] \to X$ such that

(1) $p(0) \in X_p$.
(2) $p((0,1]) \subset X_q$.

Let $L(p,q)$ be the space of exit paths over from p to q. Similarly, given $p < q < l$, we can define higher exit paths $\Delta_2 \to X$ between three exit paths from over (p,q), (q,l), and (p,l). They witness compositions of exit paths.

Note that we have a span:

$$X_p \xleftarrow{s} L(p,q) \xrightarrow{t} X_q \tag{3.6}$$

Now we can define the exit path category:

Definition 3.7 (HA)**.** Let $X \to P$ be a stratified space, then there is an ∞ category $\mathcal{E}xit(X)$ defines as follows:

(1) The space of objects of X is $\sqcup X_p$ for each $p \in P$.
(2) The space of morphisms from X_p to X_q is given by $L(p,q)$.
(3) The higher morphisms are given by higher paths.

$\mathcal{E}xit(X)$ has a canonical conservative map down to P.

Remark 3.8. $\mathcal{E}xit(X)$ remembers all the homotopical information of X. Intuitively, this is saying that we can glue back X, starting from the stratums X_p and glue pairs of them by the link $L(p,q)$, and then triplets of them by the higher links...

A precise statement is the following: we can define a ∞ category of stratified spaces up to stratified homotopy equivalence. By [**?**], it is equivalent to the category of ∞ categories with a conservative functors to P. The map takes stratified X to $\mathcal{E}xit(X) \to P$. The inverse takes such a category and reconstruct X from its stratums and the links.

Remark 3.9. In the case where X is a manifold and the stratification is nice, then $L(p,q)$ is the homotopically equivalent to the sphere $S(\nu)$ of the normal bundle ν of X_p inside $\bar{X}_q$.

Just like the local system case, we have the following theorem:

Theorem 3.10 (HA). *We have an equivalence of categories:*

$$Constr_C(X) \simeq Fun(\mathcal{E}xit(X), C) \tag{3.11}$$

While this is great, we will use the [**?**]'s approach, where we view $\mathrm{Constr}_C(X)$ itself as stratified category over P. Since we will not give the precise statement here:

Theorem 3.12 ([**?**]). *A constructible sheaf $\mathcal{F}$ on stratified space X is equivalent to the following data:*

(1) *A local system $\mathcal{F}_{X_p}$ on each p.*
(2) *For $p \le q$, we have the a compatibility map over the link $L(p,q)$, via the maps $s : L(p,q) \to X_p$, $t : L(p,q) \to X_q$:*

$$s^*\mathcal{F}_{X_p} \to t^*\mathcal{F}_{X_q}. \tag{3.13}$$

This means that we give a family of maps $\mathcal{F}_x \to \mathcal{F}_y$ for every exit path from $x \to y$.
(3) *Higher coherences over higher links.*

Intuitively, this is saying that just as how we can recover X from $\mathcal{E}xit(X)$, we can also recover $\mathcal{F}$ from the the local systems $\mathcal{F}_{X_p}$ and the gluing between them.

Now let X be the stratified space of field theories, and $\mathcal{F}$ be the constructible sheave of (IR limit) of QFTs. Then we see that this theorem tell us that to understand $\mathcal{F}$, we need to

(1) the local systems over each stratum. In physics speak, this is how the system over a single point on the stratum, as well as how the system change under adiabatic variation of the parameters.
(2) A compatibility map between the theories over the link: RG interfaces between the theories.

Let's try to apply these ideas!

4. Stratification in Quantum Mechanics: Berry phase and Borel-Weil

To start off, let's do this in quantum mechanics. Here a quantum theory is simply a Hilbert space together with a Hamiltonian H. The low energy limit is simply taking the groundstate space. When we study them in a family over X , we take the family of Hilbert space to be $H \times X \to X$ and define a family of Hamiltonian H_λ over the parameter space X. At each point $\lambda \in X$, let $\mathcal{E}_x \subset \mathcal{H}$ be the subspace of groundstates. Let's stratify X be the dimension of the groundstates, then $\mathcal{E}$ is a constructible sheaf over vector spaces over X, by the adiabatic theorem in Quantum mechanics. The holonomy is given by the Berry connection, which is the covariant derivative on X inherited by being a subbundle of the flat bundle H.

Let's consider a simple G-equivariant version: Take a (unitary) finite dimensional G representation V, which we take to be a Hilbert space. Let the parameter space $X = \mathfrak{g}$. Let's consider a family of Hamiltonians H_λ, defines as the following:

$$H_\lambda(v) := \lambda v \tag{4.1}$$

where the right hand side is the canonical lie algebra action. Note that this action is G-equivariant, in particular, G acts on the parameter space. Let's do a specific example:

Example 4.2 (Spin $\frac{1}{2}$ in magnetic field.). Let $G = \mathfrak{su}_2$, $V = \underline{2}$ the spin $\frac{1}{2}$ representation. Then we can identify $\mathfrak{g} \simeq \mathbb{R}^3$ with the lie algebra action being the cross product, we will write the vector as $\vec{B}$. Furthermore, x, y, z acts on V by the pauli matricies X, Y, Z, which we can write as $\vec{S}$. Therefore the Hamiltonian is

$$H_{\vec{B}} := \vec{B} \cdot \vec{S} = B_x X + B_y Y + B_z Z \tag{4.3}$$

Viewing B as the magnetic field and V the internal spin degrees of freedom, then this is the spin-magnetic coupling.

The stratification of $\mathbb{R}^3$, as mentioned above, is simply $0 \to \mathbb{R}^3$. That is, away from 0, where the stabilizer subgroup is $SU(2)$, the stabilizers subgroup is a maximal torus of rotation away from that vector. Note that the orbit is $SU(2)/U(1) = S^2 \simeq \mathbb{C}P^1$ is the flag variety [2]

Let's look at the groundstates bundle $\mathcal{E}$. In the generic stratum $\vec{B} \neq 0$, the hermitian matrix $\vec{B}\vec{S}$ is non-zero and has distinct eigenstates. One easy way to see that use the G symmetry to rotate so $\vec{B} = kz$, so

$$\vec{B} \cdot \vec{S} = \begin{pmatrix} k & 0 \\ 0 & -k. \end{pmatrix} \tag{4.4}$$

We pick the $-k$ eigenstates. This defines a line bundle $\mathcal{E}$ over $\mathbb{R}^3 - 0$. Let's study this at an orbit $\mathbb{C}P^1$. Note that $\mathbb{C}P^1$ has a holomorphic structure, and with a little more work we can show that $\mathcal{E}$ forms a holomorphic line bundle over the flag variety, in fact it is the line bundle $\mathcal{O}(-1)$!

Remark 4.5. There is a connection on that line bundle, it is called the Berry connection.

This is not surprising as $\mathcal{E} \subset \mathbb{C}^2 \times \mathbb{C}P^1$. Note that $\mathbb{C}^2$, the doublet is percisely $\mathbb{E}|_0$! Instead of starting with the doublet, it is straightforward to generalize:

Proposition 4.6. *Given simple G and V an irreducible representation of G, then over the generic stratum, the associated constructible sheaves restricts to a line bundle $\mathcal{E}$. Restrict further to an G orbit $\mathcal{B} = G/T$, $\mathcal{E}$ is a holomophic line bundle on G/T. In fact, the holomorphic section (of the dual of) $\mathcal{E}$ is V.*

Thus this gives a concrete simple construction of the Borel-Weil line bundle! While we recover this theorem, the interpretation/intuition is quite different.

Here's the first difference: Let's recall that in Borel-Weil, we construct the line bundle n G/B using parabolic induction. Here the line bundle is simple: first we embed $\mathcal{B}$ as a generic orbit in $\mathfrak{g}$, for a fix $\lambda \in \mathfrak{g}$, we take the eigenvector with the lowest eigenvalue. This is also the lowest weight vector of the T_λ acting on V, where T_λ isthe maximal torus generated by λ. As λ varies in, the groundstate varies and forms a nontrivial line bundle.

In particular, we actually define the vector bundles in parabolic Borel-Weil all at once: for a nongeneric λ, whose stabilizer is a parabolic subgroup $P \subset G$, with orbit partial flag variety G/P. Then λ's lower eigenstates might have degeneracy and this defines a vector bundle on G/P. However, its holomorphic sections are still E. They are all nicely glued together into a single constructible sheaf over $\mathfrak{g}$.

[2]one again we are doing compact mod maximal torus, which is equivalent to the complex mod Borel.

Remark 4.7. It is important that we work with the real G because the $\mathfrak{g}$ action is (anti)-hermitian and we can order the eigenvalues to defines what the lowest one is. For $G_{\mathbb{C}}$ the eigenvalues are just complex, which doesn't have any ordering.

Here's the second difference: In Borel-Weil, we recover V from the line bundle $\mathbb{E}$ by taking holomorphic sections. However, here, the relation between V and $\mathcal{E}_{G/B}$ is different. Recall that V is the value of $\mathcal{E}$ over the most singular stratum 0. Recall that the data of a constructible sheaf is the local system over each stratum aka the Borel-Weil line (really vector) bundles, together with the compatibility over the link for for each two pairs of stratum p, q.

Let's take p to be the origin and q the generic stratum, which is homotopic to $\mathcal{B} = G/B$, one can show that the homotopy type of the link $L(p, q)$ is homotopic to G/B too, with the span diagram:

$$* \xleftarrow{id} G/B \longrightarrow G/B \tag{4.8}$$

In general, it looks like

$$G/P' \longleftarrow P'\backslash G/P \longrightarrow G/P \tag{4.9}$$

Over the link, we need a map $t^*\mathcal{E}_{|G/B} = \mathcal{E}_{|G/B} \to s^*\mathcal{E}_* = V$. This is describing how when we get the ground states near the origin, how they are embedded into the degenerate ground state at the origin. The nontriviality of the stalk at the origin is coming from the fact the twisting over the Borel-Weil line bundles. That is, the curvature is protecting the ground state degeneracy at the diabolical point. Furthermore, the diabolical version describes parabolic induction in a geometric manner, by having them as coadjoint stratums and considering the link between them.

Remark 4.10. There is a generalization of this to infinite-dimensional $U(\mathfrak{g})$ modules and Beilinson-Bernstein. We will just remark that the D-module structures on G/B is coming from the Berry connection. Once again we can think about it as asking for the lowest weights over T_λ for each λ.

The main point here is that, even in quantum mechanics, we can find interesting mathematics.

5. Symmetry Breaking phases and anomalies in QFT

Let's kick it up a notch by considering higher dimension quantum systems, that is, Quantum field theories. I will complete this part later.

1 Oxford St, Cambridge, MA 02139
Email address: `yuleonliu@math.harvard.edu`