

Erdős #819

REFLECTED SIDON LOWER BOUND

YU LEON LIU

ABSTRACT. We sharpen the lower bound for Erdős Problem #819: writing $f(N) := \max\{|(A+A) \cap [1, N]| : A \subseteq \{1, \dots, N\}, |A| = \lfloor \sqrt{N} \rfloor\}$, we prove $\liminf_{N \rightarrow \infty} f(N)/N \geq (16\sqrt{2} - 17)/12 \approx 0.4690$, improving the classical Erdős–Freud bound of $3/8 = 0.375$ and closing most of the gap to the trivial upper bound $f(N)/N \leq 1/2 + o(1)$. The construction is a reflected two-copy of an asymptotically maximum Sidon set, obtained by thinning a Bose–Chowla set, with a small uniform random shift; the proof uses Pikhurko’s uniformity lemma.

Erdős Problem #819 [1] concerns the maximum size of the sumset of a square-root-density subset of $[1, N]$:

Definition 1. For $N \geq 1$, set

$$f(N) := \max\left\{|(A+A) \cap [1, N]| : A \subseteq \{1, \dots, N\}, |A| = \lfloor N^{1/2} \rfloor\right\}.$$

Since $|A+A| \leq \binom{|A|+1}{2}$ for any set A , we have the trivial upper bound

$$f(N) \leq \binom{\lfloor \sqrt{N} \rfloor + 1}{2} \leq N/2 + O(\sqrt{N}).$$

The classical lower bound, due to Erdős and Freud [2],

$$(2) \quad f(N) \geq \left(\frac{3}{8} - o(1)\right) N,$$

is given by an explicit construction (a union of an arithmetic progression and a structured complement).

In this note we sharpen the lower bound:

Theorem 3. *We have*

$$\liminf_{N \rightarrow \infty} \frac{f(N)}{N} \geq \frac{16\sqrt{2} - 17}{12} \approx 0.4690.$$

The proof outline is as follows: the construction is a reflected two-copy of a maximum Sidon set with some displacement, for the subsequence $N = 4q^2$ with q a sufficiently large positive integer. We control the overlap of the Sidon set and its reflected copy by Pikhurko’s uniformity lemma [3, Lemma 10] for asymptotically maximum Sidon sets, which is the core technical statement of this note. Using random-shift optimization, we get a closed form formula (21) for the expected score, which we optimize (Corollary 25) to obtain the bound. Lastly, an interpolation argument (proof of Theorem 3) lifts from the subsequence $N = 4q^2$

to all sufficiently large N . This construction and proof strategy are inspired by those of Pikhurko [3, Lemma 12].

We start the proof by recalling the notion of a Sidon set:

Definition 4. A finite set S is a *Sidon set* if all ordered sums $a + b$ with $a, b \in S$ are distinct up to the unordered pairs $\{a, b\}$.

The core technical input we need is the following uniformity statement due to Pikhurko [3, Lemma 10]. It is a common generalization of an earlier result of Erdős–Freud [2, Lemma 1] (uniformity in subintervals) and of Lindström [6] (uniformity in residue classes).

Lemma 5 (Pikhurko’s uniformity lemma). *Let q be large and let $S \subset [0, M]$ be an asymptotically maximum Sidon set of size q , so $M = (1 + o(1))q^2$. Fix an integer $m \geq 1$. Then, uniformly in subintervals $I \subseteq [0, M]$ and in residue classes $\ell \pmod{m}$,*

$$|S \cap I \cap (\ell + m\mathbb{Z})| = \frac{|I|}{mq} + o(q),$$

where the $o(q)$ depends only on m and on $q \rightarrow \infty$.

Remark 6. Conceptually, S has density $\approx 1/q$ in $[0, M]$, and the lemma says this density is preserved on every macroscopic subinterval and every residue class simultaneously, with additive error $o(q)$. (This is the form of Pikhurko’s [3, Lemma 10] statement obtained by the substitution $n = M$, $\sqrt{n} = q$.)

We will need such asymptotically maximum Sidon sets to exist for every large q (not merely along a sparse subsequence such as prime powers). This is classical:

Lemma 7 (existence of asymptotically maximum Sidon sets). *For every sufficiently large integer q , there exists a Sidon set $S \subset [0, M]$ with $|S| = q$ and $M = (1 + o(1))q^2$.*

Proof. Let p be the smallest prime with $p \geq q$; by the prime number theorem, $p = (1 + o(1))q$. The Bose–Chowla construction [5] produces a Sidon set $S^* \subset [0, p^2 - 2]$ with $|S^*| = p$. Take any subset $S \subseteq S^*$ with $|S| = q$ and set $M := \max S$; any subset of a Sidon set is Sidon, so S is Sidon. Then $M \leq p^2 - 2 = (1 + o(1))q^2$. Conversely, the classical upper bound for Sidon sets in $[0, M]$ (Erdős–Turán; see also Lindström [6]), applied to S , gives

$$q \leq M^{1/2} + O(M^{1/4}),$$

and hence $M \geq (1 - o(1))q^2$. □

We now consider the sumset and difference set of S :

Definition 8. We write

$$\Sigma := S + S, \quad \Delta := S - S.$$

By the Sidon property,

$$|\Sigma| = \binom{q+1}{2} = \frac{q(q+1)}{2}, \quad |\Delta| = q(q-1) + 1.$$

Lemma 5 allows us to give a density description of Σ and Δ on macroscopic windows. Recall that $N = 4q^2$, so $2M/N \rightarrow 1/2$. Let us define the triangular densities:

$$(9) \quad \phi(y) := \begin{cases} y, & 0 \leq y \leq 1/2, \\ 1 - y, & 1/2 \leq y \leq 1, \\ 0, & \text{otherwise,} \end{cases} \quad \psi(y) := \max(0, 1 - 2|y|).$$

We illustrate ϕ and ψ below:

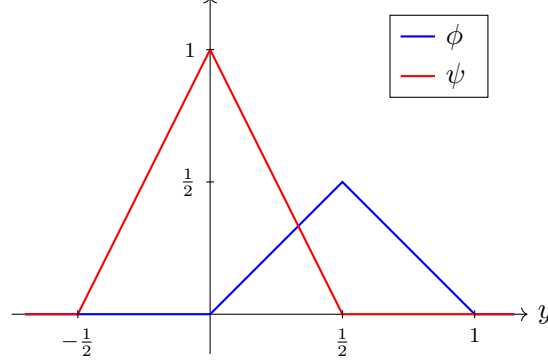


FIGURE 1. The triangular densities ϕ (blue) and ψ (red).

We now derive density descriptions for Σ and Δ on macroscopic windows.

Lemma 10. *Let q be large, and $S \subset [0, M]$ be an asymptotically maximum Sidon set of size q , so $M = (1 + o(1))q^2$, and $N = 4q^2$. Fix $\eta > 0$ and $x \in \mathbb{R}$, and let $J \subset \mathbb{Z}$ be the integer interval $[[x \cdot N/2], [x \cdot N/2] + \lfloor \eta N/2 \rfloor]$. It has size $|J| = \lfloor \eta N/2 \rfloor$. Then for any fixed integer $m \geq 1$ and any residue class $\ell \pmod{m}$,*

$$\frac{|(S + S) \cap J \cap (\ell + m\mathbb{Z})|}{|J|/m} = \phi(x) + O(\eta) + o_q(1),$$

$$\frac{|(S - S) \cap J \cap (\ell + m\mathbb{Z})|}{|J|/m} = \psi(x) + O(\eta) + o_q(1).$$

Here the implicit constant in $O(\eta)$ is absolute (i.e., independent of the other parameters), and $o_q(1)$ is a quantity that, for each fixed $\eta > 0$ and m , tends to 0 as $q \rightarrow \infty$ uniformly in $x \in \mathbb{R}$ and $\ell \pmod{m}$.

Proof. We first prove the assertion for $\Sigma = S + S$. Put $L := N/2$, so that $M = (1/2 + o(1))L$. Fix $0 < \rho \ll \eta^2$, and partition $[0, M]$ into half-open intervals I_i of length ρL , except possibly for one final shorter interval. The number of intervals is $O_\rho(1)$, and Lemma 5 gives, uniformly in i and in each residue class $r \pmod{m}$,

$$|S \cap I_i \cap (r + m\mathbb{Z})| = \frac{|I_i|}{mq} + o(q).$$

Let

$$T_\ell(J) := \#\{(a, b) \in S^2 : a + b \in J, a + b \equiv \ell \pmod{m}\}.$$

For each pair of boxes I_i, I_j , summing over residue classes $r_i + r_j \equiv \ell \pmod{m}$ gives

$$\sum_{\substack{r_i, r_j \pmod{m} \\ r_i + r_j \equiv \ell}} |S \cap I_i \cap (r_i + m\mathbb{Z})| \cdot |S \cap I_j \cap (r_j + m\mathbb{Z})| = \frac{|I_i| |I_j|}{m q^2} + o(q^2),$$

uniformly in i, j . Summing over the boxes for which $I_i + I_j \subseteq J$ gives a lower bound for $T_\ell(J)$, while summing over the boxes for which $I_i + I_j$ meets J gives an upper bound. The difference between the two box sums is supported on a boundary strip of width $O(\rho L)$ and contributes $O((\rho/\eta)|J|) = O(\eta|J|)$ by our choice $\rho \ll \eta^2$.

Thus the box count is a Riemann-sum approximation to

$$\frac{1}{m q^2} \cdot \text{area} \{(s, t) \in [0, M]^2 : s + t \in J\}.$$

The cross-section length of the square $[0, M]^2$ at $s + t = yL$ is $L\phi(y) + o(L)$ uniformly in y , since $M = (1/2 + o(1))L$. Therefore, since J has normalized length $\eta + o(1)$ and ϕ is Lipschitz,

$$\text{area} \{(s, t) \in [0, M]^2 : s + t \in J\} = L |J| (\phi(x) + O(\eta) + o_q(1)).$$

Using $q^2 = N/4 = L/2$, we obtain

$$T_\ell(J) = \frac{2|J|}{m} (\phi(x) + O(\eta) + o_q(1)).$$

Now pass from ordered to unordered pairs. Each unordered pair $\{a, b\}$ with $a \neq b$ contributes two ordered pairs; the diagonal $a = b$ contributes at most $q = o(|J|)$. Hence the number of unordered pairs $\{a, b\} \subset S$ with $a + b \in J$ and $a + b \equiv \ell \pmod{m}$ is

$$\frac{|J|}{m} (\phi(x) + O(\eta) + o_q(1)).$$

By the Sidon property, unordered pairs give distinct sums, so

$$|\Sigma \cap J \cap (\ell + m\mathbb{Z})| = \frac{|J|}{m} (\phi(x) + O(\eta) + o_q(1)),$$

which proves the first estimate.

The proof for $\Delta = S - S$ is the same with the ordered-pair count

$$U_\ell(J) := \#\{(a, b) \in S^2 : a - b \in J, a - b \equiv \ell \pmod{m}\}.$$

The corresponding continuous cross-section length at $a - b = yL$ is $L(1/2 - |y|)_+ + o(L)$, so after dividing by $q^2 = L/2$ the density is $\psi(y) = \max(0, 1 - 2|y|)$. Hence

$$U_\ell(J) = \frac{|J|}{m} (\psi(x) + O(\eta) + o_q(1)).$$

The map $(a, b) \mapsto a - b$ is injective on $\{a \neq b\}$ by the Sidon property; the zero difference contributes q ordered pairs but only one element of Δ , an $O(q) = o(|J|)$ discrepancy. Therefore

$$|\Delta \cap J \cap (\ell + m\mathbb{Z})| = \frac{|J|}{m} (\psi(x) + O(\eta) + o_q(1)).$$

□

We now turn to the construction, which is probabilistic; we derandomize at the end. Fix any $u \in (0, 1/4)$ and any $\eta \in (0, u/2)$. Let $S \subset [0, M]$ be an asymptotically maximum Sidon set of size q with $M = (1 + o(1))q^2$, and $N = 4q^2$. Let a, b be independent integers chosen uniformly at random from $\{1, \dots, \lfloor \eta N/2 \rfloor\}$. Define

$$B := a + S, \quad C := N/2 + \lfloor uN/2 \rfloor - b - S, \quad A := B \cup C.$$

Observation 11. Note that

$$B \subset [1, (\frac{1}{2} + \eta + o(1))\frac{N}{2}], \quad C \subset [(\frac{1}{2} + u - \eta - o(1))\frac{N}{2}, (1 + u)\frac{N}{2}].$$

Both are subsets of $[1, N]$, so $A \subset [1, N]$. Moreover, B and C are disjoint for q sufficiently large:

$$(12) \quad \max B \leq (\frac{1}{2} + \eta + o(1))\frac{N}{2} < (\frac{1}{2} + u - \eta - o(1))\frac{N}{2} \leq \min C$$

since $\eta < u/2$. Hence $|A| = |B| + |C| = q + q = 2q = \sqrt{N}$.

Observe that $A + A = (B + B) \cup (B + C) \cup (C + C)$. We compute the pointwise densities:

Lemma 13. *Uniformly for $v \in [1, N]$, with $x := 2v/N$,*

$$(14) \quad \Pr(v \in B + B) = \phi(x) + O(\eta) + o_q(1),$$

$$(15) \quad \Pr(v \in C + C) = \phi(2 + 2u - x) + O(\eta) + o_q(1),$$

$$(16) \quad \Pr(v \in B + C) = \psi(x - 1 - u) + O(\eta) + o_q(1),$$

$$(17) \quad \Pr(v \in (B + B) \cap (B + C)) = \phi(x)\psi(x - 1 - u) + O(\eta) + o_q(1),$$

$$(18) \quad \Pr(v \in (B + C) \cap (C + C)) = \psi(x - 1 - u)\phi(2 + 2u - x) + O(\eta) + o_q(1).$$

Moreover, for q sufficiently large (depending on η and u), $(B + B) \cap (C + C) = \emptyset$; in particular, $\Pr(v \in (B + B) \cap (C + C)) = 0$ and $\Pr(v \in (B + B) \cap (B + C) \cap (C + C)) = 0$.

Proof. Let $h := \lfloor \eta N/2 \rfloor$. We first prove (14)–(16).

For $B + B$,

$$v \in B + B \iff v - 2a \in \Sigma.$$

As a ranges over $\{1, \dots, h\}$, the quantity $v - 2a$ ranges over an interval of length $2h + O(1) = \eta N + O(1)$, but only through one fixed parity class. Therefore we apply Lemma 10 to Σ with modulus $m = 2$ and window length $2h$. The factor of 2 in the window length cancels the factor of $1/2$ from the parity restriction. Since the left endpoint of this window differs from v by $O(\eta N)$, the Lipschitz continuity of ϕ gives

$$\Pr(v \in B + B) = \phi(x) + O(\eta) + o_q(1).$$

Similarly,

$$v \in C + C \iff N + \lfloor uN \rfloor - v - 2b \in \Sigma$$

(up to an $O(1)$ shift from the floor). Again b gives a window of length $2h + O(1)$ in one fixed parity class, so Lemma 10 with $m = 2$ gives

$$\Pr(v \in C + C) = \phi(2 + 2u - x) + O(\eta) + o_q(1).$$

For $B + C$, write $t := N/2 + \lfloor uN/2 \rfloor$. Then

$$v \in B + C \iff v - t - a + b \in \Delta.$$

For each fixed a , as b varies over $\{1, \dots, h\}$, the quantity $v - t - a + b$ ranges over an interval of length $h + O(1)$ with no parity restriction. Hence Lemma 10 applied to Δ with $m = 1$, together with $a/(N/2) = O(\eta)$ and Lipschitz continuity of ψ , gives

$$\Pr(v \in B + C \mid a) = \psi(x - 1 - u) + O(\eta) + o_q(1)$$

uniformly in a . Averaging over a gives (16).

Disjointness of $B + B$ and $C + C$ for q large. We have $\max(B + B) = 2 \max B \leq (1 + 2\eta + o(1))N/2$ while $\min(C + C) = 2 \min C \geq (1 + 2u - 2\eta - o(1))N/2$. Since $\eta < u/2$ strictly, the gap $2u - 4\eta > 0$ is fixed; for q large enough that the $o(1)$ correction is smaller than this gap, $\max(B + B) < \min(C + C)$ strictly.

Pair-overlap factorizations (17) and (18). The event $\{v \in B + B\}$ is determined by a alone. Conditional on $a = a_0$, the event $\{v \in B + C\}$ becomes $v - t - a_0 + b \in \Delta$, which is again a single window query for Δ of length h in b . By Lemma 10 (with $m = 1$), uniformly in a_0 ,

$$\Pr(v \in B + C \mid a = a_0) = \psi(x - 1 - u) + O(\eta) + o_q(1),$$

because the a_0 shift of the window center is $a_0/(N/2) = O(\eta)$. Therefore

$$\Pr(v \in (B + B) \cap (B + C)) = \Pr(v \in B + B) \cdot (\psi(x - 1 - u) + O(\eta) + o_q(1)),$$

and combining with (14) (and absorbing products of error terms into $O(\eta) + o_q(1)$) gives (17).

The proof of (18) is identical with a and b interchanged: conditional on $b = b_0$, the event $\{v \in C + C\}$ is determined by b_0 , while the conditional probability of $\{v \in B + C\}$ is again $\psi(x - 1 - u) + O(\eta) + o_q(1)$. Together with (15), this gives (18). \square

Remark 19. Lemma 13 says that the pointwise densities of $B + B$, $B + C$, $C + C$ and their pairwise intersections are approximately constant (up to error $O(\eta) + o_q(1)$) on macroscopic windows.

We are now ready for our main theorem:

Theorem 20. *With the random construction above and $N = 4q^2$, for any $u \in (0, 1/4)$ and $\eta \in (0, u/2)$ we have*

$$\mathbb{E}(|(A + A) \cap [1, N]|) \geq \frac{N}{2} \cdot F(u) - O(\eta N) - o(N),$$

where

$$(21) \quad F(u) := 1 - 2u^2 - \frac{(1 - 2u)^3}{12}.$$

Proof. By Observation 11 we have $A \subset [1, N]$ and $|A| = 2q$. For q sufficiently large, $(B + B) \cap (C + C) = \emptyset$ by Lemma 13, and a fortiori the triple intersection also vanishes. Applying inclusion-exclusion to $A + A = (B + B) \cup (B + C) \cup (C + C)$,

$$\begin{aligned} \Pr(v \in A + A) &= \Pr(v \in B + B) + \Pr(v \in C + C) + \Pr(v \in B + C) \\ &\quad - \Pr(v \in (B + B) \cap (B + C)) - \Pr(v \in (B + C) \cap (C + C)). \end{aligned}$$

By Lemma 13, uniformly in v ,

$$\Pr(v \in A + A) = (p + d + r - pd - dr)(x) + O(\eta) + o_q(1),$$

where we use the shorthand:

$$p(x) = \phi(x), \quad d(x) = \psi(x - 1 - u), \quad r(x) = \phi(2 + 2u - x).$$

Summing over $v \in [1, N]$ yields a Riemann sum in $x = 2v/N \in [0, 2]$,

$$(22) \quad \mathbb{E}(|(A + A) \cap [1, N]|) \geq \frac{N}{2} \int_0^2 (p + d + r - pd - dr) dx - O(\eta N) - o(N).$$

It remains to compute those integrals for $u \in (0, 1/4)$. Recall that the area under ϕ on $[0, 1]$ is $\int_0^1 \phi(y) dy = 1/4$, and the area under ψ on $[-1/2, 1/2]$ is $\int_{-1/2}^{1/2} \psi(y) dy = 1/2$. Hence

$$\int_0^2 p dx = \int_0^1 \phi(y) dy = \frac{1}{4}, \quad \int_0^2 d dx = \int_{-1/2}^{1/2} \psi(y) dy = \frac{1}{2}.$$

For r , use the substitution $y = 2 + 2u - x$:

$$\int_0^2 r dx = \int_{2u}^{2+2u} \phi(y) dy = \int_0^1 \phi(y) dy - \int_0^{2u} y dy = \frac{1}{4} - 2u^2,$$

valid since ϕ vanishes outside $[0, 1]$ and $2u \leq 1/2$. Thus

$$(23) \quad \int_0^2 (p + d + r) dx = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} - 2u^2 = 1 - 2u^2.$$

For the overlaps, pd is supported on $\text{supp}(\phi(x)) \cap \text{supp}(\psi(x - 1 - u)) = [1/2 + u, 1]$, where $\phi(x) = 1 - x$ and $\psi(x - 1 - u) = 2x - (1 + 2u)$. With the substitution $y = x - (1/2 + u)$,

$$\int_0^2 pd dx = \int_0^{1/2-u} ((1 - 2u)y - 2y^2) dy = \frac{(1/2 - u)^3}{3} = \frac{(1 - 2u)^3}{24}.$$

By the reflection symmetry $x \mapsto 2(1 + u) - x$ (which swaps p with r while fixing d , hence swaps pd with dr while preserving the integral),

$$(24) \quad \int_0^2 pd dx = \int_0^2 dr dx = \frac{(1 - 2u)^3}{24}.$$

Combining (23) and (24),

$$\int_0^2 (p + d + r - pd - dr) dx = 1 - 2u^2 - \frac{(1 - 2u)^3}{12} = F(u).$$

Substituting into (22) gives the displayed expectation bound. \square

We now optimize u to maximize $F(u)$:

Corollary 25. *The maximum of $F(u)$ on $(0, 1/4)$ is achieved at $u^* = \frac{3}{2} - \sqrt{2} \approx 0.0858$, with value*

$$F(u^*) = \frac{16\sqrt{2}-17}{6}.$$

Setting $c := \frac{F(u^*)}{2} = \frac{16\sqrt{2}-17}{12} \approx 0.4690$, by Theorem 20 we have

$$(26) \quad \mathbb{E}(|(A + A) \cap [1, N]|) \geq N \cdot c - O(\eta N) - o(N).$$

Proof. $F'(u) = -4u + \frac{(1-2u)^2}{2} = \frac{4u^2-12u+1}{2}$. The roots are $u = \frac{3 \pm 2\sqrt{2}}{2} = \frac{3}{2} \pm \sqrt{2}$; only $u^* = \frac{3}{2} - \sqrt{2} \in (0, 1/4)$, and $F''(u) = -4 - 2(1 - 2u) < 0$ on this interval, so u^* is the unique maximum. Direct substitution gives the displayed value of $F(u^*)$. \square

Figure 2 illustrates the three density profiles p, d, r at the three values $u = 0$, $u = u^* = \frac{3}{2} - \sqrt{2}$, and $u = 1/4$. At $u = 0$ the supports of d and r overlap heavily with p and d respectively (large overlap defects, low score). At $u = 1/4$ the support of r is pushed past $x = 2$ and gets truncated (also low score). The optimum $u = u^*$ balances these two failure modes.

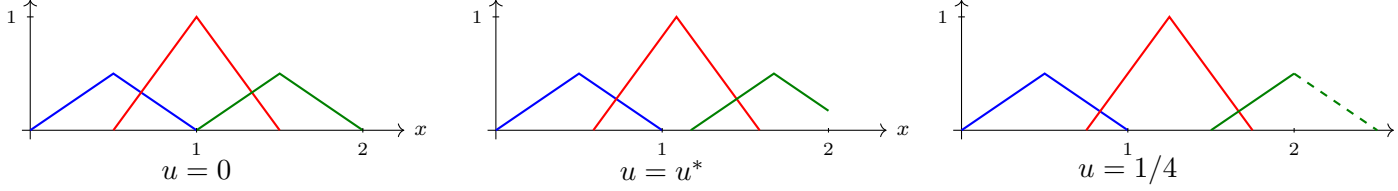


FIGURE 2. The three local density profiles p, d, r on $[0, 2]$ at $u = 0$, $u = u^* = \frac{3}{2} - \sqrt{2} \approx 0.0858$, and $u = 1/4$. At $u = u^*$ the support of r just begins to spill past $x = 2$ (small truncation); at $u = 1/4$ the truncation is substantial (dashed portion).

We are now ready to prove Theorem 3.

Proof of Theorem 3. We first establish the bound

$$(27) \quad \liminf_{N \rightarrow \infty} \frac{f(N)}{N} \geq c \approx 0.4690$$

along the subsequence $N = 4q^2$. Fix any $\eta \in (0, u^*/2)$. By Corollary 25, for q large,

$$\mathbb{E}(|(A + A) \cap [1, N_q]|) \geq N_q \cdot c - O(\eta N_q) - o(N_q),$$

where the implicit constant in $O(\eta)$ is absolute and $o(N_q)$ vanishes as $q \rightarrow \infty$ (for fixed η). Pick, for each q , a realization (a, b) achieving at least the expectation; this produces deterministic sets $A_q^{(\eta)} \subset [1, N_q]$ with $|A_q^{(\eta)}| = 2q$ and

$$|(A_q^{(\eta)} + A_q^{(\eta)}) \cap [1, N_q]| \geq c \cdot N_q - O(\eta N_q) - o(N_q).$$

Dividing by N_q and taking $\liminf_{q \rightarrow \infty}$ gives, for every $\eta \in (0, u^*/2)$,

$$\liminf_{q \rightarrow \infty} \frac{f(N_q)}{N_q} \geq c - O(\eta).$$

Since the $O(\eta)$ constant is absolute, letting $\eta \downarrow 0$ yields $\liminf_{q \rightarrow \infty} f(N_q)/N_q \geq c$, which is (27) along the subsequence $N = 4q^2$.

It remains to lift this bound to all N . Given any large N , set $q := \lfloor \sqrt{N}/2 \rfloor$, so that

$$(28) \quad N_q = 4q^2 \leq N < 4(q+1)^2 = N_{q+1}, \quad N_{q+1} - N_q = 8q + 4 = O(\sqrt{N}).$$

In particular $N - N_q \leq 8q + 4 = o(N)$ and $q \rightarrow \infty$ as $N \rightarrow \infty$.

Let $A_q \subset [1, N_q]$ be any admissible competitor for $f(N_q)$ (i.e., $|A_q| = 2q$). Since $A_q \subset [1, N_q] \subseteq [1, N]$ and $\lfloor \sqrt{N} \rfloor - 2q \in \{0, 1\}$, we may extend A_q to a set $A := A_q \cup E$ with

$|A| = \lfloor \sqrt{N} \rfloor$ and $A \subset [1, N]$ by adjoining at most one extra point. Adding points only enlarges the sumset, so $|(A + A) \cap [1, N]| \geq |(A_q + A_q) \cap [1, N_q]|$. Taking the maximum over A_q gives $f(N) \geq f(N_q)$. Hence

$$\frac{f(N)}{N} \geq \frac{f(N_q)}{N} = \frac{f(N_q)}{N_q} \cdot \frac{N_q}{N} \geq (c - o_q(1)) \cdot (1 - o(1)) = c - o(1),$$

where we used the subsequence bound $\liminf_{q \rightarrow \infty} f(N_q)/N_q \geq c$ in the second inequality. Taking $\liminf_{N \rightarrow \infty}$ gives the displayed inequality. \square

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REFERENCES

- [1] T. F. Bloom, *Erdős Problem #819*, <https://www.erdosproblems.com/819>, accessed May 15, 2026. [1](#)
- [2] P. Erdős and R. Freud, *On sums of a Sidon-sequence*, J. Number Theory **38** (1991), no. 2, 196–205. [1](#), [2](#)
- [3] O. Pikhurko, *Dense edge-magic graphs and thin additive bases*, Discrete Mathematics **306** (2006), 2097–2107. [1](#), [2](#)
- [4] J. Singer, *A theorem in finite projective geometry and some applications to number theory*, Trans. Amer. Math. Soc. **43** (1938), 377–385.
- [5] R. C. Bose and S. Chowla, *Theorems in the additive theory of numbers*, Comment. Math. Helv. **37** (1962/63), 141–147. [2](#)
- [6] B. Lindström, *Well distribution of Sidon sets in residue classes*, J. Number Theory **69** (1998), 197–200. [2](#)

1 OXFORD ST, CAMBRIDGE, MA 02139
 Email address: yuleonliu@math.harvard.edu