Erdős #153: FINITE MEAN-SQUARE GAP DIVERGENCE

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Lemma 1 (ESS94–Bloom shifted-intersection bound)**.** *For every finite Sidon set A of integers with $|A| = n$ and every nonzero integer d,*

$$\big|(A + A) \cap (A + A + d)\big| \;\leq\; 2n^{3/2}.$$

Remark 2. This is implicit in the proof of [3, Theorem 2] and was extracted in finite form by Bloom [4] on the Erdős Problems #152 forum thread.

For a finite Sidon set A with $A + A = \{s_1 < \cdots < s_t\}$, write $Q(A) \coloneqq \frac{1}{t}\sum_{i=1}^{t-1}(s_{i+1} - s_i)^2$. Erdős Problem #153 quickly follows:

Theorem 3. *For every sequence of finite Sidon sets A_n with $|A_n| \to \infty$, $Q(A_n) \to \infty$.*

Proof. Let A be a Sidon set with $|A| = n$, let $T = A + A = \{s_1 < \cdots < s_t\}$, and set $d_i \coloneqq s_{i+1} - s_i$. Since A is Sidon, $t = n(n+1)/2$. Given $d > 0$, each index i with $d_i = d$ gives a distinct element $s_{i+1} = s_i + d \in T \cap (T + d)$. So by Lemma 1

$$\#\{i \colon d_i = d\} \leq |T \cap (T + d)| \leq 2n^{3/2}. \tag{4}$$

Fixing $R > 0$ and summing over $d \in \{1, \dots, R\}$, we have

$$\#\{i \colon 1 \leq d_i \leq R\} \leq 2Rn^{3/2}. \tag{5}$$

Since $2Rn^{3/2} = o(t)$ as $n \to \infty$ for fixed R, the number of gaps with $d_i > R$ is at least $(1 - o(1))\,t$, uniformly in A. Hence

$$Q(A) = \frac{1}{t}\sum_{i=1}^{t-1} d_i^2 \geq R^2(1 - o(1)) \quad (n \to \infty),$$

uniformly over Sidon sets A of size n. Applied to any sequence A_n with $|A_n| \to \infty$, this gives $\liminf_n Q(A_n) \geq R^2$; since R is arbitrary, $Q(A_n) \to \infty$. ☐

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References

[1] T. F. Bloom, *Erdős Problem #153*, https://www.erdosproblems.com/153, accessed May 2026.

[2] T. F. Bloom, *Erdős Problem #152*, https://www.erdosproblems.com/152, accessed May 2026.

[3] P. Erdős, A. Sárközy, V. T. Sós, *On sum sets of Sidon sets, I*, J. Number Theory **47** (1994), 329–347. DOI: 10.1006/jnth.1994.1040. 1

[4] T. F. Bloom, *Forum comment on Erdős Problem #152* (post 5224, 3 Apr 2026), https://www.erdosproblems.com/forum/thread/152. 1

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