Erdős #153: ASYMPTOTICALLY MAXIMUM CASE

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Erdős Problem #153 [1] asks the following question. For a finite Sidon set A with $|A| = n$, write $A + A = \{s_1 < \cdots < s_t\}$, where $t = |A + A| = \binom{n+1}{2} \sim n^2/2$. Define

$$Q(A) \;:=\; \frac{1}{t}\sum_{i=1}^{t-1}(s_{i+1} - s_i)^2.$$

Does $Q(A) \to \infty$ for every sequence of finite Sidon sets with $|A| \to \infty$?

Write $A = \{a_1 < \cdots < a_n\}$ and $D = \operatorname{diam}(A) = a_n - a_1$. Note that $s_1 = 2a_1$ and $s_t = 2a_n$, so the sumset spans an interval of length $s_t - s_1 = 2D$.

Observation 1 (Complementary regime). Cauchy–Schwarz gives $Q(A) \geq 4D^2/t^2$, so we have $Q(A) \to \infty$ whenever $D/n^2 \to \infty$ (since $t \sim n^2/2$). This was observed by Dutta [2]. It follows that we are reduced to the case $D = O(n^2)$.

In this note we give a short proof of the densest case, where $D = (1 + o(1))\, n^2$. We call such a Sidon set *asymptotically maximum.*[1]

Theorem 2. *Let $(A_n)_{n\geq 1}$ be a sequence of Sidon sets with $|A_n| = n$ and $\operatorname{diam}(A_n) = (1 + o(1))\, n^2$. Then $Q(A_n) \to \infty$.*

Remark 3. The case $\liminf \operatorname{diam}(A_n)/n^2 > 1$ remains open.

The remainder of this note is devoted to proving Theorem 2. The key input is the following sumset density description, which follows from Pikhurko's uniformity lemma [5, Lemma 10] by the same convolution argument used in [6, Lemma 10]. Define the triangular density

$$\varphi(y) \;:=\; \begin{cases} y, & 0 \leq y \leq 1/2, \\ 1 - y, & 1/2 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 4 ([6, Lemma 10])**.** *Let $A \subset [0, D]$ be an asymptotically maximum Sidon set of size n, so $D = (1 + o(1))\, n^2$. Fix $\eta > 0$ and let $J \subset [0, 2D]$ be an integer interval of length $|J| = \lfloor \eta D \rfloor$ centered at $x \cdot D$ for some $x \in (0, 2)$. Then*

$$\frac{|(A + A) \cap J|}{|J|} \;=\; \varphi(x/2) + O(\eta) + o_n(1),$$

where $o_n(1) \to 0$ as $n \to \infty$, uniformly in x.

In particular, $\varphi(y) \to 0$ as $y \to 0$ or $y \to 1$: the sumset density vanishes linearly at the boundary of $[0, 2D]$. We need the following general inequality, which lets us localize the sum of squared gaps.

Lemma 5 (Localization)**.** *Let $T = \{s_1 < \cdots < s_t\} \subset \mathbb{R}$ with gaps $d_i = s_{i+1} - s_i$, and let $I_1, \ldots, I_N \subset [s_1, s_t]$ be pairwise disjoint intervals. Then*

$$\sum_i d_i^2 \;\geq\; \sum_{k=1}^{N} \frac{|I_k|^2}{|T \cap I_k| + 1}.$$

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[1] Examples are obtained from Bose–Chowla sets [4], together with the classical Sidon upper bound [3].

Proof. For each k, the $m_k = |T \cap I_k|$ points of T in I_k cut it into $m_k + 1$ pieces; call their lengths $\ell_1^{(k)}, \dots, \ell_{m_k+1}^{(k)}$. By Cauchy–Schwarz,

$$\sum_j \big(\ell_j^{(k)}\big)^2 \;\geq\; \frac{|I_k|^2}{m_k+1}.$$

Each piece $\ell_j^{(k)}$ is contained in some gap of T of length d_i, so $\ell_j^{(k)} \leq d_i$. Since the intervals I_k are pairwise disjoint, pieces from different intervals occupy disjoint parts of the real line, and pieces lying in the same gap d_i are disjoint subintervals of that gap with total length $\leq d_i$. Writing ℓ for a generic piece length, we have

$$\sum_{\text{all pieces}} \ell^2 \;=\; \sum_i \sum_{\substack{\text{pieces}\\ \text{in gap } i}} \ell^2 \;\leq\; \sum_i \sum_{\substack{\text{pieces}\\ \text{in gap } i}} \ell \cdot d_i \;=\; \sum_i d_i \cdot \Big(\underbrace{\sum_{\substack{\text{pieces}\\ \text{in gap } i}} \ell}_{\leq d_i} \Big) \;\leq\; \sum_i d_i^2,$$

where the first inequality uses $\ell \leq d_i$ for each piece, and the last uses that the pieces in gap i are disjoint subintervals of that gap. Combining with the Cauchy–Schwarz bound above gives the result. □

Proof of Theorem 2. Since $Q(A)$ is invariant under translation, we may assume $\min A_n = 0$, so that $A_n \subset [0, D]$ with $\max A_n = D$. Write $T = A_n + A_n = \{s_1 < \cdots < s_t\}$ and $d_i = s_{i+1} - s_i$, with $D = (1 + o(1))\, n^2$ and $t \sim n^2/2$.

Fix constants $0 < \eta < \varepsilon < 1/2$, independent of n. Take $N = \lfloor (1/2 - \varepsilon)/\eta \rfloor$ consecutive disjoint intervals $I_1, \dots, I_N$ of width $\Delta = 2\eta D$, starting at $2\varepsilon D$, and ignore the leftover part. The k-th interval I_k is centered at $u_k \cdot 2D$ where $u_k = \varepsilon + (k - \frac{1}{2})\eta$. Since $u_k \leq 1/2$, we have $\varphi(u_k) = u_k$.

Since η and ε are fixed, $\Delta = 2\eta D \to \infty$ as $n \to \infty$. Applying Lemma 4 with parameter 2η, for all n sufficiently large (depending on η, ε),

$$m_k \;=\; |T \cap I_k| \;=\; (u_k + E_{k,n})\, \Delta,$$

where $|E_{k,n}| \leq C_0 \eta + o_n(1)$ uniformly in k, for an absolute constant C_0. For all n sufficiently large, $m_k + 1 \leq (u_k + 2C_0\eta)\, \Delta$, and so by Lemma 5,

$$\sum d_i^2 \;\geq\; \sum_{k=1}^{N} \frac{\Delta^2}{m_k+1} \;\geq\; \sum_{k=1}^{N} \frac{\Delta}{u_k + 2C_0\eta} \;=\; \sum_{k=1}^{N} \frac{2\eta D}{u_k + 2C_0\eta}.$$

Using $2D/t \to 4$ as $n \to \infty$ (since $D = (1 + o_n(1))\, n^2$ and $t \sim n^2/2$), for all n sufficiently large,

$$Q(A_n) \;=\; \frac{1}{t} \sum d_i^2 \;\geq\; 3\,\eta \sum_{k=1}^{N} \frac{1}{u_k + 2C_0\eta}.$$

The right-hand side depends only on η and ε (not on n). It is a Riemann sum with step size η for $\int_\varepsilon^{1/2} du/(u + 2C_0\eta)$, which converges as $\eta \to 0$ to $\int_\varepsilon^{1/2} du/u = \log(1/(2\varepsilon))$. This in turn diverges as $\varepsilon \to 0$.

Given $B > 0$, first choose $\varepsilon > 0$ so small that $3\log(1/(2\varepsilon)) > 2B$. Then choose $0 < \eta < \varepsilon$ sufficiently small so that the Riemann sum exceeds B. Finally choose n large enough that all $o_n(1)$ errors are negligible. This gives $Q(A_n) > B$ for all large n. Since B is arbitrary, $Q(A_n) \to \infty$. □

References

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