UPPER BOUNDS FOR Erdős PROBLEM #327 AND ITS k-VARIANT

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ABSTRACT. Let $k \geq 1$, and let $f_k(N)$ denote the maximum size of a subset $A \subseteq \{1, \ldots, N\}$ such that $a + b \nmid k\,a\,b$ for all distinct $a, b \in A$. Set

$$\alpha_k \;\coloneqq\; \limsup_{N\to\infty} \frac{f_k(N)}{N}.$$

We introduce a smooth–rough decomposition method that gives explicit upper bounds for α_k. For each finite set of primes P and integer cutoff $X \geq 1$, the bound reduces to a finite computation of the prefix-by-prefix maximum size of a k-admissible subset of the P-smooth integers up to X. Certified computations with $P = \{2, 3, 5, 7, 11, 13\}$ and $X = 2000$ give

$$\alpha_1 \;\leq\; 0.7769, \qquad \alpha_2 \;\leq\; 0.7630.$$

Both bounds improve van Doorn's 25/28 bound for Erdős Problem #327.

Erdős Problem #327 asks how large a subset $A \subseteq [1, N]$ can be if no two distinct elements $a, b \in A$ satisfy $a + b \mid ab$, and asks a stronger variant with $a + b \nmid 2ab$. The odd integers show that the first problem has density at least 1/2. The purpose of this note is to give explicit upper bounds for these densities using a finite computation on smooth numbers.

Definition 1. Fix an integer $k \geq 1$. A subset $A \subseteq \mathbb{N}$ is said to be *k-admissible* if

$$a + b \;\nmid\; k\,a\,b \qquad \text{for all distinct } a, b \in A.$$

Definition 2. For integers $N, k \geq 1$, set

$$f_k(N) \;\coloneqq\; \max\big\{\, |A| \;:\; A \subseteq \{1, \ldots, N\},\; A \text{ is } k\text{-admissible} \,\big\},$$

and define the asymptotic density

$$\alpha_k \;\coloneqq\; \limsup_{N\to\infty} \frac{f_k(N)}{N} \;\in\; [0, 1].$$

Observation 3. The set of odd integers is k-admissible for all odd k. It follows that $\alpha_k \geq \frac{1}{2}$ for k odd.

Erdős Problem #327 [1] asks:

> Suppose $A \subseteq \{1, \ldots, N\}$ is such that if $a, b \in A$ and $a \neq b$ then $a + b \nmid ab$. Can A be 'substantially more' than the odd numbers? What if $a, b \in A$ with $a \neq b$ implies $a + b \nmid 2ab$? Must $|A| = o(N)$?

In our notation, the first question asks whether one can substantially improve on the lower bound $\alpha_1 \geq 1/2$ furnished by the odd integers ($f_1(N) \geq \lceil N/2 \rceil$). The second question asks whether $\alpha_2 = 0$. For $k = 1$, van Doorn [2] proved the upper bound

$$\alpha_1 \;\leq\; \tfrac{25}{28} \;\approx\; 0.8929. \tag{4}$$

A trivial observation transfers this to all $k \geq 1$: if $a + b \mid ab$ then $a + b \mid k\,a\,b$, so every k-admissible set is also 1-admissible. Hence $f_k(N) \leq f_1(N)$ and

$$\alpha_k \;\leq\; \tfrac{25}{28} \;\approx\; 0.8929 \qquad \text{for all } k \geq 1. \tag{5}$$

The question whether $\alpha_2 = 0$ remains open.

We investigate upper bounds for α_k. Our main result is the following.

Theorem 6. *We have the bounds*

$$\alpha_1 \;\leq\; 0.7769, \qquad \alpha_2 \;\leq\; 0.7630.$$

This follows directly from Theorem 19 applied at $P = \{2, 3, 5, 7, 11, 13\}$ and $X = 2000$, together with the numerical computation of $\Phi_{P,k}(X)$ recorded in Table 1. The α_1 bound improves van Doorn's 25/28 bound (4), and the α_2 bound improves the inherited bound (5).

To start we use the smooth–rough decomposition:

Definition 7. Let P be a finite subset of primes. A positive integer n is P-*smooth* if all of its prime factors are in P, and it is P-*rough* if it is coprime to every prime in P. We denote by S_P the set of P-smooth integers. In particular, $1 \in S_P$ vacuously.

Then, given P, every positive integer n uniquely decomposes as $n = m \cdot c$ where c is P-smooth and m is P-rough.

Observation 8. Let $A \subseteq \mathbb{N}$ be a finite set, and for each P-rough integer m define $A_m \coloneqq \{c \in S_P : m \cdot c \in A\}$. The unique decomposition of integers into P-smooth and P-rough parts gives the disjoint partition

$$A \;=\; \bigsqcup_{m\ P\text{-rough}} m \cdot A_m. \tag{9}$$

Lemma 10. *Let $A \subseteq [1, N]$ be k-admissible and m a P-rough integer. Then A_m is also k-admissible.*

Proof. Let $c, d \in A_m$ be distinct. Then $mc, md \in A$ are distinct. If $c + d \mid k\,c\,d$, then

$$mc + md \;=\; m(c + d) \;\Big|\; m^2 \cdot k\,c\,d \;=\; k(mc)(md),$$

contradicting the k-admissibility of A. Hence $c + d \nmid k\,c\,d$, so A_m is k-admissible. □

Next we introduce a cutoff and the associated quantities.

Definition 11. Let X be a positive integer and P a finite subset of primes. We define

$$H_P(X) \;\coloneqq\; \sum_{\substack{c \in S_P \\ c \leq X}} \frac{1}{c}.$$

We also set $H_P(\infty) \coloneqq \sum_{c \in S_P} 1/c$. By the Euler product formula, we have

$$\rho_P \coloneqq \prod_{p \in P} (1 - 1/p) = (H_P(\infty))^{-1}. \tag{12}$$

We have the following estimate for P-rough numbers at most some cutoff Y:

Lemma 13. *Let P be a finite subset of primes and $Y > 0$. Then*

$$\#\{m \leq Y : m \text{ is } P\text{-rough}\} = \rho_P \cdot Y + O(2^{|P|}), \tag{14}$$

in particular, for a fixed P, the error is $O_P(1)$, hence $o(Y)$ as $Y \to \infty$.

Proof. By inclusion–exclusion,

$$\#\{m \leq Y : m \text{ is } P\text{-rough}\} \;=\; \sum_{S \subseteq P} (-1)^{|S|} \left\lfloor \frac{Y}{\prod_{p \in S} p} \right\rfloor. \tag{15}$$

Replacing each floor by its argument introduces an error of $O(1)$ in each of the $2^{|P|}$ summands, giving

$$
\begin{aligned}
\#\{m \le Y : m \text{ is } P\text{-rough}\} \;&=\; Y \sum_{S \subseteq P} \frac{(-1)^{|S|}}{\prod_{p \in S} p} + O(2^{|P|}) \\
&=\; Y \prod_{p \in P} \Bigl(1 - \tfrac{1}{p}\Bigr) + O(2^{|P|}) \\
&=\; \rho_P \cdot Y + O(2^{|P|}).
\end{aligned} \tag{16}
$$

□

Next we define the key quantity $\Phi_{P,k}$.

Definition 17. Let P be a finite subset of primes and $X \ge 1$ an integer. List the elements of $S_P \cap [1, X]$ in increasing order as $c_1 < c_2 < \cdots < c_s$ (with $c_1 = 1$), and for each $1 \le j \le s$ define

$$\beta_j \;\coloneqq\; \max\bigl\{\, |I| \;:\; I \subseteq \{c_1, \dots, c_j\},\ I \text{ is } k\text{-admissible} \,\bigr\},$$

with the convention $\beta_0 \coloneqq 0$. We then set

$$\Phi_{P,k}(X) \;\coloneqq\; \frac{\beta_s}{c_s} \;+\; \sum_{j=1}^{s-1} \beta_j \left(\frac{1}{c_j} - \frac{1}{c_{j+1}} \right). \tag{18}$$

We are now ready for the main theorem:

Theorem 19. *For every integer $k \ge 1$, finite subset of primes P, and integer cutoff $X \ge 1$, we have*

$$\alpha_k \;\le\; 1 \;-\; \rho_P \cdot \bigl(H_P(X) - \Phi_{P,k}(X)\bigr).$$

Proof. Take $N > 0$ and let $A \subseteq [1, N]$ be a k-admissible set. For every P-rough integer m, recall the set A_m from Observation 8. It is k-admissible by Lemma 10. Exchanging the order of summation,

$$|A| = \sum_{m\ P\text{-rough}} |A_m| = \sum_{m\ P\text{-rough}} \sum_{c \in A_m} 1 = \sum_{c \in S_P} r(c) = \underbrace{\sum_{\substack{c \in S_P \\ c \le X}} r(c)}_{\text{head}} \;+\; \underbrace{\sum_{\substack{c \in S_P \\ X < c \le N}} r(c)}_{\text{tail}}, \tag{20}$$

where $r(c) \coloneqq \#\{m : m \text{ is } P\text{-rough},\ cm \in A\}$.

Tail. Applying Lemma 13 with $Y = N/c$, we have $r(c) \le \rho_P(N/c) + O_P(1)$ for each c. Summed over P-smooth $c \le N$, the accumulated error is $O_P\bigl(|S_P \cap [1, N]|\bigr) = O_P\bigl((\log N)^{|P|}\bigr) = o(N)$ for fixed P, so

$$
\begin{aligned}
\sum_{\substack{c \in S_P \\ X < c \le N}} r(c) &\le \rho_P N \sum_{\substack{c \in S_P \\ X < c \le N}} \frac{1}{c} + o(N) \\
&\le \rho_P N \sum_{\substack{c \in S_P \\ X < c}} \frac{1}{c} + o(N) \\
&= \rho_P N \bigl(H_P(\infty) - H_P(X)\bigr) + o(N) \\
&= N\bigl(1 - \rho_P H_P(X)\bigr) + o(N).
\end{aligned} \tag{21}
$$

Head. Here we parametrize by m again:

$$\sum_{\substack{c \in S_P \\ c \le X}} r(c) = \sum_{m\ P\text{-rough}} |A_m \cap [1, X]|. \tag{22}$$

Define $j^*(m) := \max\{j : c_j \le \min(X, N/m)\}$, with the convention that $j^*(m) = 0$ when $N/m < c_1 = 1$ (in which case $A_m \cap [1, X] = \emptyset$, consistent with $\beta_0 = 0$). It follows that $A_m \cap [1, X]$ is a k-admissible set contained in $\{c_1, \ldots, c_{j^*(m)}\}$; the definition of β_j in Definition 17 implies that $|A_m \cap [1, X]| \le \beta_{j^*(m)}$.

Next we group m by $j^*(m)$. By Lemma 13, for $1 \le j \le s-1$ the number of P-rough m with $N/c_{j+1} < m \le N/c_j$ is $\rho_P N(1/c_j - 1/c_{j+1}) + o(N)$; for $j = s$ the number of P-rough m with $m \le N/c_s$ is $\rho_P N/c_s + o(N)$. It follows from Equation (22) that

$$\sum_{\substack{c \in S_P \\ c \le X}} r(c) \le \beta_s \rho_P N/c_s + \sum_{j=1}^{s-1} \beta_j \rho_P N\big(1/c_j - 1/c_{j+1}\big) + o(N) = \rho_P N \Phi_{P,k}(X) + o(N). \tag{23}$$

Combining. Putting the head and tail together,

$$|A| \le N\big(1 - \rho_P \cdot (H_P(X) - \Phi_{P,k}(X))\big) + o(N), \tag{24}$$

and the result follows by dividing by N and taking $N \to \infty$. □

We conclude with the finite computations used in Theorem 6. For each choice of P, X, and k, we computed the prefix maxima β_j appearing in Definition 17 and then evaluated the bound in Theorem 19. The resulting upper bounds are summarized in Table 1.

P	X	$k = 1$ bound	$k = 2$ bound	$k = 3$ bound
$\{2, 3\}$	100	0.8631	0.8631	0.6686
$\{2, 3, 5\}$	200	0.8249	0.8188	0.6440
$\{2, 3, 5, 7\}$	500	0.7990	0.7907	0.6224
$\{2, 3, 5, 7\}$	1000	0.7915	0.7830	0.6148
$\{2, 3, 5, 7, 11\}$	1000	0.7870	0.7773	0.6149
$\{2, 3, 5, 7, 11, 13, 17, 19\}$	1000	0.7895	0.7751	0.6271
$\{2, 3, 5, 7, 11, 13\}$	2000	**0.7769**	**0.7630**	**0.6089**

Table 1. Upper bounds for α_k obtained from Theorem 19. Note that the second-to-last row ($P = \{2, \ldots, 19\}, X = 1000$) gives *weaker* bounds than the bottom row ($P = \{2, \ldots, 13\}, X = 2000$), illustrating that at this scale doubling the cutoff X is more effective than adding further primes to P.

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References

[1] T. F. Bloom, *Erdős Problem #327*, https://www.erdosproblems.com/327, accessed 13 May 2026. 1

[2] W. van Doorn, Forum comment on Erdős Problem #301 (multiplicative-shift covering argument giving the 25/28 upper bound), https://www.erdosproblems.com/301, accessed 13 May 2026. 1

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